MATHEMATICS 3210-1. 3-rd Midterm Test (Sample): Solutions.

November 10, 2001

The exam is "closed book and closed notes". In your solutions you can use formulas for the derivatives of $e^x, \log(x)$, of the trigonomentric functions and of the polynomial functions. You can use that these functions are continuous on their domains.

1. (20 points) For $L, a \in \mathbb{R}$ prove the following:
$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L$ if and only if $\lim_{x\to a} f(x) = L$.
You can use ϵ-δ definition of the limit if you like.
Solution. See the textbook.

2. (15 points) Give the definition of
(a) Limit $\lim_{x\to -\infty} f(x) = L$.
(b) The derivative of a function f at $a \in \mathbb{R}$.
Solution. See the textbook.

3. (15 points) Prove that there exists $x \in \mathbb{R}$ such that $\frac{e^x - e^{-x}}{2} = \cos(x)$.

Solution. The function $f(x) = \frac{e^x - e^{-x}}{2} - \cos(x)$ is continuous. $f(0) = 0 - 1 = -1$,

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \frac{\infty - 0}{2} = \infty.$$

On the other hand, $\cos(x)$ is bounded. Thus there exists $b > 0$ so that $\frac{e^b - e^{-b}}{2} > \cos(b)$. Thus $f(b) > 0$. Hence the function f changes sign on the interval $[0, b]$. Now the intermediate value theorem implies that $f(x) = 0$ for some $x \in (0, b)$. Hence

$$\frac{e^x - e^{-x}}{2} = \cos(x). \quad \square$$

Below is a more concrete way to find b. Consider $b = 2$. Recall that $2 < e$. Thus

$$\frac{e^2 - e^{-2}}{2} \geq \frac{2^2 - 2^{-2}}{2} = \frac{4 - 0.25}{2} > 1.5$$

Since $\cos(b) < 1$, hence $f(b) > 1.5 - 1 = 0.5 > 0$. Thus we can take $b = 2$ and use the same argument as above. $\square$

4. (15 points) Compute the limit (or show that it does not exist)

$$\lim_{x\to 0+} x\cos(\frac{1}{x}).$$

Solution. Consider a sequence $x_n > 0$ which converges to zero. Then $\cos(\frac{1}{x_n})$ is a bounded sequence and the sequence x_n converges to zero. Thus the sequence $x_n \cos(\frac{1}{x_n})$ converges to zero. Hence, by the definition of the limit of a function,

$$\lim_{x\to 0+} x\cos(\frac{1}{x}) = 0.$$

5. (20 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, everywhere differentiable function. Is it true that $f'(x) \geq 0$ for each $x \in \mathbb{R}$? Give a proof!

Solution. The answer is "true". Suppose that $f'(a) < 0$ for some $a \in \mathbb{R}$. Then

$$0 > f'(a) = \lim_{x\to a+} \frac{f(x) - f(a)}{x - a}$$

implies that there exists $\delta > 0$ such that for $a < x < a + \delta$ we have:

$$\frac{f(x) - f(a)}{x - a} < 0,$$

which implies that $f(x) - f(a) < 0$ and hence $f(x) < f(a)$. This contradicts the assumption that f is strictly increasing. □

Remark. You cannot argue that "if $f'(a) < 0$ then f is strictly decreasing near a", since you do not know that f is countinuously differentiable and hence you do not know that $f'(t) < 0$ for all t near a.

6. (15 points) Determine if the series

$$\sum \frac{1}{n\log(n)}$$

converges.

Solution. Consider the integral

$$\int_e^T \frac{dx}{x\log(x)} = [\log(\log(x))]_e^T = \log(\log(T)).$$

Recall that $\lim_{T\to\infty} \log(T) = +\infty$, hence $\lim_{T\to\infty} \log(\log(T)) = +\infty$. Thus $\int_e^\infty \frac{dx}{x\log(x)} = +\infty$.

On the other hand, $\frac{1}{x\log(x)} > 0$ is a strictly decreasing function for $x > 0$ since both x and $\log(x)$ are strictly increasing. Thus we can apply the integral test to the series $\sum \frac{1}{n\log(n)}$ and conclude that this series diverges to $+\infty$. □