MATHEMATICS 3210-1. Second Midterm Test (Sample): Solutions.

October 17, 2001

The exam is "closed book, closed notes", you can use a 1 page cheat sheet with the elementary college algebra formulae. All problems should be treated as problems about "proofs"; just the correct computation without proper justification can result in a very low score on the problem.

1. (15 points) Prove or disprove the following:

For each real number x there exists a natural number n such that $n/(n^2 - n) > x$.

Solution. The inequality is clearly true for each $x \leq 0$; so consider the case $x > 0$. The inequality is equivalent to $n < 1/x + 1$. The number $n = 1$ satisfies this inequality. Hence the answer is YES.

2. (15 points) State the Bolzano-Weierstrass Theorem.

Solution. See the textbook.

3. (15 points) Compute the limit (or show that it does not exist)

$$\lim_{n\to\infty} \frac{\sqrt{n} - n}{\sqrt{n} + 2n}$$

(You can use Limit Theorems.)

Solution.

$$\frac{\sqrt{n} - n}{\sqrt{n} + 2n} = \frac{1 - \sqrt{n}}{1 + 2\sqrt{n}} =$$

Let's divide numerator and denominator by $\sqrt{n}$:

$$= \frac{\frac{1}{\sqrt{n}} - 1}{\frac{1}{\sqrt{n}} + 2}.$$

Note that $\lim \frac{1}{\sqrt{n}} = \sqrt{0} = 0$ (the square root theorem). Hence, by the ratio theorem:

$$\lim_{n\to\infty} \frac{\sqrt{n} - n}{\sqrt{n} + 2n} = \frac{-1}{2} = -0.5. \quad \square$$

4. (20 points) Let $x_1 \geq 3$. Define inductively the sequence (x_n) as $x_{n+1} = \sqrt{3x_n}$. Show that this sequence is nonincreasing. Find the limit of this sequence.

Solution. Let's verify that $x_k \geq 3$ using induction:

1. $x_1 \geq 3$ by the assumption.

2. Suppose that $x_k \geq 3$. Then $x_{k+1} = \sqrt{3x_k} \geq \sqrt{3 \cdot 3} = 3$.

Hence, by induction, $x_n \geq 3$ for each n.

Let's verify now that $x_{n+1} \leq x_n$, i.e. $\sqrt{3x_n} \leq x_n$, $\iff$

$$3x_n \leq x_n^2 \iff 0 \leq x_n(x_n - 3)$$

The latter is true since $3 \leq x_n$.

Thus $x_{n+1} \leq x_n$ for all n. Hence the sequence is nonincreasing.

Note that each $x_n \geq 3$, hence we have a nonincreasing sequence which is bounded from below. Thus there exists a limit

$$\lim x_n = a \in \mathbb{R}, a \geq 3.$$

Then

$$a = \lim x_{n+1} = \lim \sqrt{3x_n}.$$

By the square root theorem for the limits of sequences:

$$\lim \sqrt{3x_n} = \sqrt{3a}.$$

Thus $a = \sqrt{3a}$, hence $a^2 = 3a$, $a = 0$ or $a = 3$. Since we know that $a \geq 3$, we conclude that $a = 3$. Thus

$$\lim x_n = 3. \quad \square$$

5. (20 points) State and prove the theorem about the limit of sum of two convergent sequences.

See the textbook.

6. (15 points) Determine if the following series converges:

$$\sum_{i=1}^{\infty} \frac{\sin(n) + \cos(n)}{3^n}.$$

(You can use tests for convergence.)

Solution. Note that $|\cos(n) + \sin(n)| \leq 1 + 1 = 2$. Thus

$$|\frac{\sin(n) + \cos(n)}{3^n}| \leq 2(\frac{1}{3})^n.$$

Note that the series $\sum (\frac{1}{3})^n$ converges (it is a geometric series and $0 < 1/3 < 1$). Hehce $\sum 2(\frac{1}{3})^n$ converges as well. Therefore

$$\sum_{i=1}^{\infty} \frac{\sin(n) + \cos(n)}{3^n}.$$

also converges by the comparison test. $\square$