

MATHEMATICS 3210-1. First Midterm Test (Sample): Solutions.

September 11, 2001

1. Using axioms of real numbers only show that product of 0 and every real number equals zero.

Solution. For each $x \in \mathbb{R}$ we have $0 \cdot x = (0 + 0) \cdot x$; hence $0 \cdot x = 0 \cdot x + 0 \cdot x$, this $0 = 0 \cdot x$. \square

2. State the completeness axiom for the real numbers.

3. Prove or disprove that the sets

$$A = \{n \in \mathbb{Z} \mid n = -5k + 1 \text{ for some } k \in \mathbb{N}\},$$

and \mathbb{N} have the same cardinality.

Solution. Consider the map

$$f : \mathbb{N} \rightarrow A, f(x) = -5x + 1.$$

By the definition of A , f is onto. To see that f is injective note that if $-5x + 1 = -5y + 1$ then $x = y$ (subtract 1 and divide the inequality by -5). Thus f is bijective. Hence A and \mathbb{N} have the same cardinality.

4. Represent the function

$$f(x) = 1 + x^2 + x^4 + x^{2/3}$$

as a nontrivial composition of two functions.

Solution. Let $h(x) = x^2$, $g(t) = 1 + t + t^2 + t^3$. Then $f = g \circ h$.

5. Negate the following statement:

“For each $x \in \mathbb{R}$ there exists a natural number n so that either $nx > 1$ or $nx < -1$.”

Solution. The negation is:

“There exists $x \in \mathbb{R}$ such that for each natural number n we have $-1 \leq nx \leq 1$.”

6. Show that for each $n \in \mathbb{N}$

$$n! \geq 2^n - 2.$$

Solution. Let's verify this via induction. For $n = 1$ the inequality reads

$$1 \geq 2 - 2 = 0$$

which is a true inequality.

Step of induction: suppose that $k! \geq 2^k - 2$. We want to show that

$$(k + 1)! \geq 2^{k+1} - 2.$$

By the induction hypothesis,

$$(k + 1)! = (k + 1) \cdot k! \geq (k + 1)(2^k - 2).$$

Thus to show that $(k + 1)! \geq 2^{k+1} - 2$ it suffices to prove

$$(k + 1)(2^k - 2) \geq 2 \cdot 2^k - 2,$$

equivalently,

$$2^k(k - 1) \geq 2k, \text{ or } 2^{k-1}(k - 1) \geq k.$$

The last inequality is true for $k = 1$. For $k \geq 2$ the inequality follows from the valid inequalities

$$2^{k-1} \geq 2 \geq \frac{k}{k-1}.$$

(Note that $2(k - 1) \geq k$ for all $k \geq 2$.) □