Math 3210-2. 3-rd Midterm Test: Solutions.

1. (15 points) State the definition of uniformly continuous function. Give example of a function which is continuous but not uniformly continuous. Justify the example!

Solution. f is uniformly continuous on a set S if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in S$ satisfy $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. Consider the function $f(x) = \frac{1}{x}$ on $(0, \infty)$. Then f is continuous (since each rational function is continuous on its domain). Let $x_n = \frac{1}{n} \in (0, \infty)$. Then (x_n) is Cauchy since it converges to zero. On the other hand, $f(x_n) = n$ diverges to $+\infty$, hence it is not Cauchy. Since for each absolutely continuous function f, the $f(x_n)$ is Cauchy provided that x_n is Cauchy, we conclude that $f(x) = \frac{1}{x}$ is not Cauchy. □

2. (20 points) Prove that if a function f is differentiable at a then f is continuous at a.

Solution. See the textbook.

3. (15 points) Show that $e^x > x$ for all real numbers $x \geq 0$.

Solution. $(e^x)' = e^x > 0$. Hence the function e^x is strictly increasing on $[0, \infty)$.

Consider the function $f(x) = e^x - x$. Then $f(0) = 1 - 0 = 1 > 0$. Let's prove that f is increasing on $[0, \infty)$. For $x > 0$, $f'(x) = e^x - 1 > 1 - 1 = 0$. Hence $f'(x) > 0$ on $(0, \infty)$ and so the function f is strictly increasing. Thus $f(x) > f(0) > 0$ for each $x \in (0, \infty)$. Hence $3^x > x$ for all real numbers $x \geq 0$. □

4. (15 points) Determine if the limit

$$\lim_{x \to -\infty} x(\sin(x) + 2)$$

exists and compute this limit if it exists. You can use limit theorems if you like.

Solution. Let $x_n \to -\infty$ be a sequence of real numbers. Recall that $\sin(x) \geq -1$ for all x, hence $\sin(x) + 2 \geq 1$ for all x. Thus $x_n(\sin(x_n) + 2) \geq x_n$. Hence (by sandwich lemma)

$$\lim_{n \to \infty} x_n(\sin(x_n) + 2) \geq \lim_{n \to \infty} x_n = -\infty.$$

Therefore

$$\lim_{x \to -\infty} x(\sin(x) + 2) = -\infty. \quad \square$$

5. (15 points) Determine if the function

$$f(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x > 0 \\ x^3 & \text{if } x \leq 0 \end{cases}$$

is differentiable at zero and compute the derivative if it exists. You can use limit theorems if you like.

Solution. Consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}.$$

To compute this limit first consider

$$\lim_{x \to 0+} \frac{f(x)}{x} = \lim_{x \to 0+} \frac{x^2 \cos(1/x)}{x} = \lim_{x \to 0+} x \cos(1/x) = 0$$

since $\cos(1/x)$ is bounded and $\lim_{x \to 0+} x = 0$.

Now consider

$$\lim_{x \to 0-} \frac{f(x)}{x} = \lim_{x \to 0-} \frac{x^3}{x} = \lim_{x \to 0-} x^2 = 0.$$

Hence the limit from the left equals the limit from the right, thus $f'(0)$ exists and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0. \quad \square$$

6. (20 points) Show that the function $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$, admits a continuous inverse which is defined on the whole interval $[0, \infty)$. (You can use any theorem about continuous functions you like.)

Solution. The function x^2 is continuous, hence the image of f is an interval J. Let's find the lower and upper bounds of this interval. $f(0) = 0 \leq x^2$ for each x, hence $0 \in J$ is the lower bound for J.

$$\lim_{x \to \infty} x^2 = +\infty,$$

hence the upper bounds for J is $+\infty$. Hence $J = [0, \infty)$.

The function f is strictly increasing on $[0, \infty)$ since for $y > x \geq 0$ we have $y^2 > x^2$. Thus (by the inverse function theorem) f admits a continuous inverse function defined on whole interval $[0, \infty)$.