MATHEMATICS 3210-1. First Midterm Test: Solutions.

September 11, 2001

1. (15 points) Negate the statement:

For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$, one has

$$|\frac{\sin(x)}{x} - 1| \leq \epsilon.$$

(Your negation should be *nontrivial*: it should not involve things like "not for every", or "there does not exist".)

Solution. The statement has the form:

$\forall \epsilon > 0 \exists \delta > 0 | \forall x \in (-\epsilon, \epsilon) \backslash \{0\}$ satisfy A.

Here A is the inequality:

$$|\frac{\sin(x)}{x} - 1| \leq \epsilon.$$

Thus the negation should have the form

$\exists \epsilon > 0 | \forall \delta > 0 \exists x \in (-\epsilon, \epsilon) \backslash \{0\}$ so that A is false.

In English, the negation is:

There exists $\epsilon > 0$ so that for each $\delta > 0$ there exists $x \in (-\epsilon, \epsilon) \backslash \{0\}$ so that

$$|\frac{\sin(x)}{x} - 1| > \epsilon.$$

□

2. (20 points) Prove or disprove the following:

The function $f : \mathbb{Z} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x > -1 \\ 1/x & \text{if } x \leq -1 \end{cases}$$

is injective.

Solution. The function is injective. To verify this assume that $f(x) = f(y)$ for some $x, y \in \mathbb{Z}$. We will show that $x = y$. There are 3 cases to consider:

Case 1. $x, y > -1$. Since x, y are integers this means that $x, y \geq 0$. Thus we have $x^2 + 1 = y^2 + 1$, which impies that $x^2 = y^2$, thus $|x| = |y|$. However, since both x, y are nonnegative, we conclude that $x = y$.

Case 2. $x, y \leq -1$. Then $f(x) = f(y)$ implies that $1/x = 1/y$, thus by multiplying this equation by xy we get $y = x$.

Case 3. $x \leq -1, y \geq 0$. Then $f(x) = f(y)$ implies that $1/x = y^2 + 1$. However, $y \geq 0$ implies $y^2 + 1 \geq 1$. On the other hand, since $x < -1 < 0$, $1/x < 0$. Thus, $1/x$ is negative and $y^2 + 1$ is positive. Contradiction, hence the case 3 simply cannot occur.

Hence we proved that $f(x) = f(y)$ implies that $x = y$. Thus f is injective. $\square$

3. (15 points) When do the sets X and Y have the same cardinality? (State the definition.)

Solution. X and Y have the same cardinality if there is a bijection $f : X \to Y$. $\square$

4. (15 points) Using only the axioms of real numbers show uniqueness of the real number $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ one has $xa = 0$.

Solution. We will prove that $a = 0$ (this implies that a is unique).

Proof. Since $xa = 0$ for each $x \in \mathbb{R}$, we can take $x = 1$ and get:

$$x = x \cdot 1 = 0.$$

Hence $x = 0$. $\square$

5. (20 points) Prove that $n! \geq 2n$ for all natural numbers $n > 2$.

Solution. We use the induction.

1. For $n = 3$ the assertion $3! \geq 2 \cdot 3$ is true (both sides are equal to 6).

2. Assume that $k! \geq 2k$ for some $k \geq 3$. We want to show that $(k + 1)! \geq 2k$.

The inequality $k! \geq 2k$ implies that $k!(k + 1) \geq 2k(k + 1)$. On the other hand, $k \geq 3$ implies that $k \geq 1$; hence

$$(k + 1)! = k!(k + 1) \geq 2k(k + 1) \geq 2(k + 1).$$

By transitivity of the $\geq$ we get: $(k + 1)! \geq 2(k + 1)$. $\square$

6. (15 points) Compute infimum of the set

$$S = \{x^2 + 1 | x \in \mathbb{R}\}$$

or show that the infimum does not exist. Explain your solution!

Solution. I claim that $\inf(S) = 1$. I will prove it by showing that $\min(S) = 1$. Indeed, $1 = 0^2 + 1 \in S$. Since $x^2 \geq 0$ for each $x \in \mathbb{R}$ (proven in the class), $x^2 + 1 \geq 1$. Thus, $1 \leq S$. This proves that $1 = \min(S)$. Recall that if $\min(S)$ exists, then $\min(S) = \inf(S)$. Thus $\inf(S) = 1$. $\square$