MATHEMATICS 3210-1. Homework 8: Solutions.

November 7, 2001

1. Section 15,

15.1. Determine which of the following series converges:

(a) $\sum \frac{(-1)^n}{n}$.

Solution. This series satisfies the alternating series test since $1/n$ is a decreasing sequence which converges to zero. $\square$

(b) $\sum \frac{(-1)^n n!}{2^n}$.

Solution. This series does not converge. The reason is that for each $n \geq 2$

$$|\frac{(-1)^n n!}{2^n}| \geq \frac{1}{2}$$

and hence the sequence $\frac{(-1)^n n!}{2^n}$ does not converge to zero. $\square$

15.4. Determine which of the following series converges:

(a) $\sum \frac{1}{\sqrt{n}\log(n)}$.

Solution. I claim that this series diverges. To prove this let's show that

$$\sqrt{x} \geq \log(x), \text{ for all } x \geq 4.$$

Indeed, this inequality is satisfied for $x = 4$ (since $e^2 \geq 4$). Comparing the derivatives of the left hand side and right hand side we get:

$$\frac{1}{2\sqrt{x}} \geq \frac{1}{x} \iff 2 \leq \sqrt{x} \iff 4 \leq x.$$

It follows that $\sqrt{x} \geq \log(x)$, for all $x \geq 4$. Thus for all $n \geq 4$ we have

$$\frac{1}{\sqrt{n}\log(n)} \geq \frac{1}{(\sqrt{n})^2} = \frac{1}{n}.$$

The series $\sum \frac{1}{n}$ diverges, hence by the comparison test, the series $\sum \frac{1}{\sqrt{n}\log(n)}$ diverges as well. $\square$

(b) $\sum \frac{\log(n)}{n}$.

Since for each $n \geq 3$ we have $\log(n) \geq 1$, it follows that $\frac{\log(n)}{n} \geq \frac{1}{n}$. Hence by the comparison test, the series $\sum \frac{\log(n)}{n}$ diverges. $\square$

(c) $\sum \frac{1}{n(\log(n))\log(\log(n))}$.

Solution.
$$\int \frac{1}{x(\log(x))\log(\log(x))}dx = \log(\log(\log(x))).$$
Since $\lim_{n\to\infty} \log(\log(\log(n))) = \infty$ we conclude that by the integral test, the series $\sum \frac{1}{n(\log(n))\log(\log(n))}$ diverges. □

(d) $\sum \frac{\log n}{n^2}$.

Solution. Using the result of (a) we see that $\log(n) \leq \sqrt{n}$ for $n \geq 4$. Hence
$$\frac{\log n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$
Since $3/2 > 1$ we see that the series $\sum \frac{1}{n^{3/2}}$ converges. Thus by the comparison test, the series $\sum \frac{\log n}{n^2}$ also converges. □

15.6.

(a) Give example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

Solution. Take $a_n = \frac{1}{n}$. □

(b) Show that if $\sum a_n$ is a convergent series of nonnegative terms then $\sum a_n^2$ also converges.

Solution. Since $\sum a_n$ converges, $\lim a_n = 0$. Hehce for all large n, $a_n < 1$. Hence for large n, $a_n^2 < a_n$. Thus the series $\sum a_n^2$ converges by the comparison test. □

(c) Give example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Solution. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum a_n$ converges by the alternating series sets. However $a_n^2 = \frac{1}{n}$, hence $\sum a_n^2$ diverges. □

2. Section 17, # 17.3. I will consider only one of these examples.

(a) Prove that $f(x) = \log(1 + \cos^4 x)$ is continuous.

Solution. We are given that $\cos(x)$ is continuous. Hence by the product rule for the continuous functions $\cos^4 x$ is also continuous. Since the constant function 1 is continuous, by the sum rule for the continuous functions, $1+\cos^4 x$ is continuous. The function f is the composition of $\log(y)$ and $1+\cos^4 x$. Both functions are continuous, hence the function f is also continuous (theorem 17.5).

17.12. (a) Let f be a continuous function with domain (a, b). Suppose that $f(r) = 0$ for each rational number $r \in (a, b)$. Prove that $f(x) = 0$ for each x.

Solution. Density of rational numbers implies that for each $x \in (a, b)$ there is a sequence of rational numbers x_n such that $\lim x_n = x$. Thus, for all large n, $a < x_n < b$. Therefore (by discarding a finite number of the members of the sequence x_n we can assume that $x_n \in (a, b)$. Then $f(x_n) = 0$ (since $x_n \in \mathbb{Q}$). Thus, continuity of f implies that
$$f(x) = \lim f(x_n) = \lim 0 = 0. \quad \square$$

(b) Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for all irrational numbers. Prove that h is continuous only at $x = 0$ and is discontinuous at all other points.

Solution. First note that $|h(x)| \leq |x|$ (indeed, if $x \in \mathbb{Q}$ then $0 = |h(0)| \leq |x|$ and if x is irrational then $|h(x)| = |x|$). Let x_n be a sequence convergent to zero. Thus by the sandwich lemma we have:
$$\lim h(x_n) = \lim |x_n| = 0 = h(0).$$

Hence $\lim h(x_n) = h(0)$ and the function h is continuous at zero.

Suppose that $x \neq 0$. Case 1. x is rational. Then, by the density of irrational numbers, there exists a sequence of irrational numbers x_n convergent to x. Then

$$\lim h(x_n) = \lim x_n = x.$$

On the other hand, $h(x) = 0 \neq x$. Hence $h(x) \neq \lim h(x_n)$ and thus h is discontinuous at x.

Case 2. x is irrational. Pick a sequence of rational numbers x_n convergent to x and repeat the argument from case 1 to show that h is discontinuous at x. □

17.14. For each rational number x write x as p/q where p, q are integers with no common factors and $q > 0$. Define $f(x) = 1/q$. Define $f(x) = 0$ for each irrational number x. Show that f is continuous at each point of $\mathbb{R}\backslash\mathbb{Q}$ and discontinuous at each point $x \in \mathbb{Q}$.

Solution. First, suppose that $x \in \mathbb{Q}$. Then $f(x) = 1/q \neq 0$. On the other hand, there is a sequence of irrational numbers x_n convergent to x. Then

$$\lim f(x_n) = 0 \neq f(x).$$

Hence f is discontinuous at x.

Second, suppose that x is irrational. Suppose that x_n is a sequence convergent to x. Let A denote the set of those n's for which $x_n \in \mathbb{Q}$. Consider the subsequence $(x_n)_{n\in A}$, $x_n = p_n/q_n$. The limit of this subsequence is still x. I claim that $\lim q_n = \infty$. If not then there is an infinite subsequence $x_{n_k}, n_k \in A$ such that q_{n_k} is bounded, so we can assume that $q_{n_k} = q$ for all k. Then p_{n_k}'s should be bounded as well and hence, the sequence x_n contains an infinite subsequence which consists of equal rational numbers. Hence $x \in \mathbb{Q}$ which is not the case. Thus $\lim q_n = \infty$ which implies that $\lim_{n\in A} f(x_n) = \lim 1/q_n = 0 = f(x)$. Let B be the set of those n's for which x_n is irrational. Since $\lim_{n\in A} f(x_n) = 0$, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, n \in A$ we have

$$|f(x_n)| < \epsilon.$$

However for all $n \geq n_0, n \in B$ we also have

$$|f(x_n)| = 0 < \epsilon.$$

Thus for all $n \geq n_0$ we have $|f(x_n)| < \epsilon$. This means that $\lim f(x_n) = 0 = f(x)$. Thus the function f is continuous at the irrational number x. □

3. Section 18,

18.4. Let $S \subset \mathbb{R}$ and suppose that there exists a sequence $x_n \in S$ which converges to $a \notin S$. Prove that there exists an unbounded continuous function on S.

Solution. Let $f(x) = \frac{1}{x-a}$. Then the sequence $f(x_n) = \frac{1}{x_n - a}$ diverges to ∞, hence the function f is unbounded on S. On the other hand, f is continuous at each $x \neq a$, thus it is continuous on S. □

18.9. Prove that a polynomial function f of odd degree has at least one real root.

Proof. Let $f(x) = a_0 + a_1 x + ... + a_d x^d$ where d is odd and $a_d \neq 0$. By dividing f by a_n we can assume that $a_d = 1$, so

$$f(x) = a_0 + a_1 x + ... + a_{d-1} x^{d-1} + x^d.$$

Then

$$\lim \frac{f(n)}{n^d} = \frac{a_0 + a_1 n + ... + a_{d-1} n^{d-1} + n^d}{n^d} = 1.$$

Thus

$$\lim_{n\to\infty} f(n) = \lim n^d \frac{f(n)}{n^d} = \lim n^d = \infty.$$

Similarly,

$$\lim \frac{f(-n)}{(-n)^d} = \frac{a_0 + a_1(-n) + ... + a_{d-1}(-n)^{d-1} + (-n)^d}{(-n)^d} = 1.$$

Note that (since d is odd)

$$\lim(-n)^d = -\lim n^d = -\infty.$$

Thus

$$\lim_{n\to\infty} f(-n) = \lim(-n)^d \frac{f(-n)}{(-n)^d} = \lim(-n)^d = -\infty.$$

Therefore, there exists $a < 0$ such that $f(a) < 0$ and there is $b > 0$ such that $f(b) > 0$. Hence by the intermediate value theorem there exists $x \in (a, b)$ such that $f(x) = 0$. □