MATHEMATICS 3210. Homework # 7: Solution.

1. (a) Let $x_n = \cos(\frac{n\pi}{3})$. (b) Find a convergent subsequence and compute $\limsup x_n$.

Solution. (a) Recall that $\cos(x + 2n\pi) = \cos(x)$. Thus

$$x_{6n} = \cos(\frac{6n\pi}{3}) = \cos(2n\pi) = 1.$$

This subsequence is contstant and converges to 1.

(b) If $n = 6k + r$ where $0 \leq r < 6$ then

$$x_{6k+r} = \cos(2k\pi + \frac{r\pi}{3}) = \cos(\frac{r\pi}{3}).$$

The latter takes values 1 if $r = 0$; $1/2$, if $r = 1$; $-1/2$, if $r = 2$; 0 if $r = 3$; $-1/2$, if $r = 4$; $1/2$ if $r = 5$. The largest of these numbers equals 1. For each $N \in \mathbb{N}$ there is a natural number k such that $6k \geq N$. Hence

$$\sup\{x_n | n \geq N\} = 1.$$

Thus

$$\limsup x_n = \lim_{N\to\infty} \sup\{x_n | n \geq N\} = 1.$$

□

2. Determine wheather or not the sequence

$$x_n = \frac{n + n(-1)^n}{n+1}$$

contains a convergent subsequence.

Solution. Let's show that this sequence is bounded:

$$|x_n| = \frac{|n + n(-1)^n|}{n+1} \leq \frac{n + n|(-1)^n|}{n+1} = \frac{n+n}{n+1} = \frac{2n}{n+1}.$$

We would like to show that the sequence $\frac{2n}{n+1}$ is bounded. One way to prove it is to note that this sequence converges:

$$\lim \frac{2n}{n+1} = \lim \frac{2}{1 + 1/n} = 2.$$

(By the "fraction" limit theorem.) Since each convergent sequence is bounded we see that there exists $C > 0$ such that

$$\frac{2n}{n+1} \leq C$$

Thus $|x_n| \leq \frac{2n}{n+1} \leq C$. So the sequence (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence. □

14.5 (a). Suppose that $\sum a_n, \sum b_n$ converge. Prove that $\sum(a_n + b_n)$ converge and

$$\sum(a_n + b_n) = \sum a_n + \sum b_n.$$

Solution. Follows immediately from the "sum" theorem for the limnits of sequences.

14.7. Show that if $\sum a_n$ converges and $a_n \geq 0$ for all n then for each $p > 1$ the series $\sum a_n^p$ also converges.

Solution. Since a_n converges, $\lim a_n = 0$. Hence there exists $N \in \mathbb{N}$ such that $a_n < 1$ for each $n \geq N$. For each $0 \leq x < 1$ we have $x^p \leq x$. Hence for $n \geq N$ we have:

$$0 \leq a_n^p \leq a_n.$$

Thus $\sum a_n$ converges by the comparison test. □

14.1. (a) Determine if the following sequences converges:
$\sum \frac{n^4}{2^n}$.

Solution. Let's apply the ratio test to this sequence:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4 2^n}{2^{n+1} n^4} = \frac{1}{2}(\frac{n+1}{n})^4.$$

By the "sum" limit theorem

$$\lim \frac{n+1}{n} = \lim(1 + 1/n) = 1.$$

By the product limit theorem,

$$\lim \frac{1}{2}(\frac{n+1}{n})^4 = \frac{1}{2}1^4 = \frac{1}{2} < 1.$$

Since this limit is < 1, by the ratio test, the series converges.

Problem. Determine if the series converges:

$$\sum 2 + \cos(n)4^n.$$

Solution. Let's apply the fraction test to this sequence:

$$\frac{a_{n+1}}{a_n} = \frac{4^n(2+\cos(n+1))}{4^{n+1}(2+\cos(n))} = \frac{1}{4}\frac{2+\cos(n+1)}{2+\cos(n)}$$

We would like to show that

$$\limsup \frac{1}{4}\frac{2+\cos(n+1)}{2+\cos(n)} \leq \frac{3}{4} < 1.$$

Indeed,

$$\frac{1}{4}|\frac{2+\cos(n+1)}{2+\cos(n)}| = \frac{3}{4}.$$

Thus $\limsup \frac{1}{4}\frac{2+\cos(n+1)}{2+\cos(n)} \leq 3/4$. Hence by the ratio test, the series converges. □