

MATHEMATICS 3210. Homework # 6: Solution.

10.7. Let S be a bounded nonempty subset of \mathbb{R} and suppose $a = \sup S \notin S$. Show that there exists an increasing sequence $s_n \in S$ such that $\lim s_n = a$.

Solution. Given $n \in \mathbb{N}$ we can find (by the approximation property for the supremum) $x_n \in S$ such that $a - \frac{1}{n} \leq x_n < a$. Then by the sandwich theorem, $\lim x_n = a$. However the sequence x_n might not be increasing. We will try to find an increasing subsequence x_{n_k} in x_n .

We use the induction. Let $s_1 = x_{n_1} = x_1$. Suppose that $s_k = x_{n_k}$ was chosen so that

$$x_{n_{k-1}} < x_{n_k}.$$

Recall that $x_{n_k} < a$ (since $a \notin S$). Let n be so large that $x_{n_k} < x_n < a$ (it exists since $\lim x_n = a$). Then take $s_{k+1} = x_{n_{k+1}} := x_n$.

Thus we constructed via induction an increasing subsequence s_k is constructed so that $s_k \in S$, (s_k) is increasing and converges to a . \square

10.9. Let $s_1 = 1, s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

a) Find s_2, s_3, s_4 .

Solution. $s_2 = \frac{1}{2}, s_3 = \frac{2}{3} \frac{1}{4} = \frac{1}{6}, s_4 = \frac{3}{4} \frac{1}{36} = \frac{1}{48}$.

This suggests that the sequence is decreasing.

b) Show that $\lim s_n$ exists.

Solution. Let's verify that the sequence s_n is decreasing, i.e., $s_{n+1} < s_n$. Use the induction:

1. For $n = 1$ the inequality $s_2 < s_1$ is true since $0.5 < 1$.

2. Suppose that $s_k < s_{k+1}$. Let's show that $s_{k+1} < s_{k+2}$. The last is equivalent to

$$s_{k+1} = \left(\frac{k}{k+1}\right)s_k^2 < \left(\frac{k+1}{k+2}\right)s_{k+1}^2 = s_{k+2}.$$

Since $s_k < s_{k+1}$ it suffices to check that

$$\frac{k}{k+1} < \frac{k+1}{k+2}.$$

The latter is equivalent to

$$k(k+2) < (k+1)^2 \iff k^2 + 2k < k^2 + 2k + 1.$$

The last inequality is clearly true. Thus $s_{k+1} < s_{k+2}$.

Hence, by induction, $s_{n+1} < s_n$ for all n . Next note that each s_n is a positive number ($s_1 > 0$ and if $s_n > 0$ then $s_{n+1} = (\frac{n}{n+1})s_n^2 > 0$ as well). Thus the sequence is decreasing and bounded from below. Thus, by the theorem about convergence of monotone sequences, this sequence converges. Since the sequence is decreasing, $\lim s_n < s_1 = 1$.

c) Prove that $\lim s_n = 0$.

Proof. Let $\lim s_n = a$. We know that $a < 1$. We have:

$$a = \lim s_{n+1} = \lim \frac{n}{n+1} s_n^2$$

By applying a “fraction” limit theorem to the sequence $\frac{n}{n+1}$ we get:

$$\lim \frac{n}{n+1} = \lim \frac{1}{1+1/n} = 1.$$

By applying the product limit theorem to s_n we get:

$$\lim s_n^2 = a^2.$$

Thus by the product limit theorem: $a = \lim \frac{n}{n+1} s_n^2 = a^2$. Hence $a = a^2$, $a(a-1) = 0$. Thus $a = 1$ or $a = 0$. Since $a = 1$ is excluded by $a < 1$, we get: $a = 0$. Thus $\lim s_n = 0$. \square

10.10. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

a) Find s_2, s_3, s_4 .

Solution. $s_2 = \frac{1}{3}(1 + 1) = 2/3$. $s_3 = \frac{1}{3}(1 + 2/3) = \frac{5}{9}$. $s_4 = \frac{1}{3}(1 + 5/9) = \frac{14}{81}$. This suggests that the sequence is decreasing.

b) Use induction to show that $s_n > 1/2$ for all n .

Solution. $1 = s_1 > 0.5$. Suppose $s_k > 1/2$. Then

$$s_{k+1} = \frac{1}{3}(s_k + 1) > \frac{1}{3}(0.5 + 1) = \frac{1}{2}.$$

Thus $s_{k+1} > 1/2$ and we are done by induction.

c) Show that (s_n) is decreasing.

Solution. We want to show that $\frac{1}{3}(s_n + 1) = s_{n+1} < s_n$. Equivalently, $s_n + 1 < 3s_n$, $\iff 1 < 2s_n \iff s_n > 1/2$. The latter as we proved in (b) is true. Hence $s_{n+1} < s_n$. \square

d) Show that s_n converges and compute the limit.

Solution. The sequence is decreasing and is bounded from below. Hence, by the theorem about convergence of monotone sequences, (s_n) converges. Let $\lim s_n = a$. Then

$$a = \lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) = \frac{1}{3}(a + 1).$$

Thus $3a = a + 1$, $a = 1/2$. Hence the sequence converges to $1/2$. \square

4. Let $x_n = n(-1)^n$. Compute $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$.

Solution. $\sup\{x_n | n \geq N\} = \sup\{2k | 2k \geq N\} = \infty$. Thus

$$\limsup x_n = \lim \infty = \infty.$$

Similarly,

$$\inf\{x_n | n \geq N\} = \inf\{-(2k+1) | 2k+1 \geq N\} = -\infty.$$

Thus $\liminf x_n = -\infty$.

5. Let $x_n = (-1)^n + 1/n$. Compute $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$.

Solution. $\sup\{x_n | n \geq N\} = \sup\{1/n + 1 | n = 2k \geq N\}$, it equals $1 + 1/N$ if N is even and $1 + 1/(N+1)$ if N is odd. Thus

$$1 \leq 1 + \sup\{x_n | n \geq N\} \leq 1 + 1/N.$$

Therefore by the sandwich theorem,

$$\limsup x_n = 1.$$

Similarly, $\liminf x_n = -1$.

6. Suppose that (x_n) is a Cauchy sequence such that $x_n \in \mathbb{Z}$ for each n . Show that there exists $n_0 \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that $x_n = z$ for all $n \geq n_0$.

Proof. Since (x_n) is Cauchy, for $\epsilon = 0.5$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| \leq 0.5$ for each $n, m \geq n_0$. However $x_n, x_m \in \mathbb{Z}$, thus $x_n = x_m = x_{n_0}$ for all $n, m \geq n_0$. Thus, by taking $z = x_{n_0}$ we get the desired conclusion. \square