MATHEMATICS 3210. Homework 4: Solutions.

September 27, 2001

In all the problems you should use the definition of limit, no *limit theorems!*

1. §7, # 7.4. Give examples of (a) sequence of irrational numbers convergent to a rational number.

(b) sequence of rational numbers convergent to an irational number.

Solution. (a) Let $x_n = \sqrt{2}/n$. Then $x_n \notin \mathbb{Q}$ but $\lim_{n\to\infty} x_n = 0 \in \mathbb{Q}$.

(b) Let $a = \sqrt{2}, b_n = \sqrt{2} + 1/n$. Then density of rational numbers implies that each interval (a, b_n) contains a rational number x_n. Then

$$\lim_{n\to\infty} x_n = a = \sqrt{2}.$$

2. §8, # 8.1.

(a) Prove that $\lim_{n\to\infty}(-1)^n/n = 0$.

Proof. Let $\epsilon > 0$. The by Archimedian Principle there exists $n_0 \in \mathbb{N}$ such that $n_0 > 1/\epsilon$. Hence for all $n \geq n_0$ we have $n > 1/\epsilon$; and thus $1/n < \epsilon$. Thus $|(-1)^n/n| = 1/n < \epsilon$ for all $n \geq n_0$. Hence $\lim_{n\to\infty}(-1)^n/n = 0$.

(b) Prove that $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

Proof. Let $\epsilon > 0$. We want to find n_0 such that for ann $n \geq n_0$ we have $\frac{1}{n^{1/3}} < \epsilon$. Equivalently, $1/\epsilon < n^{1/3}$. The latter is equivalent to $1/\epsilon^3 < n$. By Archimedian Principle there exists $n_0 \in \mathbb{N}$ such that $n_0 > 1/\epsilon^3$. Hence for all $n \geq n_0$ we have $n > 1/\epsilon^3$, equivalently, $\frac{1}{n^{1/3}} < \epsilon$. Hence $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

(c) Prove that $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$.

Proof. Let $\epsilon > 0$. We want to find n_0 such that for all $n \geq n_0$ we have

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon.$$

The latter is equivalent to

$$|\frac{3(2n-1) - 2(3n+2)}{3(3n+2)}| < \epsilon \iff$$

$$|\frac{-7}{3(3n+2)}| < \epsilon \iff$$
$$\frac{7}{3(3n+2)} < \epsilon \iff$$
$$\frac{7}{3\epsilon} < 3n+2 \iff$$
$$\frac{7}{9\epsilon} - \frac{2}{3} < n.$$

By Archimedian Principle there exists $n_0 \in \mathbb{N}$ such that

$$\frac{7}{9\epsilon} - \frac{2}{3} < n_0.$$

Hence for all $n \geq n_0$ we have

$$\frac{7}{9\epsilon} - \frac{2}{3} < n.$$

The latter implies that

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon.$$

Hence $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$.

(d) Prove that $\lim_{n\to\infty} \frac{n+6}{n^2-6} = 0$.

Proof. Let $\epsilon > 0$. We want to find n_0 such that for all $n \geq n_0$ we have

$$|\frac{n+6}{n^2-6}| < \epsilon.$$

Note that for $n \geq 3$ we have $n^2 \geq 6$, thus it suffices to find $n_0 \geq 3$ so that for all $n \geq n_0$ we have

$$\frac{n+6}{n^2-6} < \epsilon.$$

The latter is equivalent to

$$(n+6)\epsilon^{-1} < n^2 - 6 \iff$$
$$6(\epsilon^{-1}+1) < n(n-\epsilon^{-1}).$$

By Archimedian Principle there exists $n_1 \in \mathbb{N}$ such that $n_1 > \epsilon^{-1} + 1$, hence for all $n \geq n_1$ we have $n - \epsilon^{-1} \geq 1$. By Archimedian Principle there exists $n_2 \in \mathbb{N}$ such that $n_2 > 6(\epsilon^{-1}+1)$. Hence for all $n \geq n_2$ we have $n > 6(\epsilon^{-1}+1)$.

Let $n_0 = \max(3, n_1, n_2)$. Then for all $n \geq n_0$ we have:

$$6(\epsilon^{-1}+1) < n \text{ and } 1 \leq (n-\epsilon^{-1}).$$

The last two inequalities imply that

$$6(\epsilon^{-1}+1) < n(n-\epsilon^{-1}).$$

Thus for all $n \geq n_0$ we have:

$$\frac{n+6}{n^2-6}| < \epsilon.$$

Hence $\lim_{n\to\infty} \frac{n+6}{n^2-6} = 0$. □

3. a) Prove that the sequence $\{(-1)^n\}$ contains subsequences which converge and subsequences which do not converge.

b) Find a convergent subsequence of $x_n = n + (-1)^{3n} n$.

c) Prove that $\lim_{n\to\infty} \pi - 3/\sqrt{n} = \pi$.

Solution. (a) By choosing $n = 2k$ (the even natural numbers) we get the subsequence $\{(-1)^{2k}\} = \{1\}$. This is a constant sequence, therefore it converges.

As we proved in the class, the sequence $\{(-1)^n\}$ does not have a limit. This sequence is a subsequence of itself, therefore we get a subsequence which does not converge.

b) Let $n = 2k + 1$ be an odd number, then $3n$ is again an odd number. (Indeed, $3n = 2n + n$. The number $2n$ is even and the number n is odd. The sum of an even and odd number is odd, thus $2n + n$ is odd.) Thus for odd n we have $(-1)^{3n} = -1$. Hence the subsequence corresponding to odd indices equals $x_{2k+1} = n - n = 0$. This is a constant sequence, therefore it converges (to zero in this case). □

c) Given $\epsilon > 0$ we want to find n_0 such that for all $n \geq n_0$ we have

$$\frac{3}{\sqrt{n}} = |\pi - 3/\sqrt{n} - \pi| < \epsilon.$$

The latter inequality is equivalent to

$$\frac{3}{\epsilon} < \sqrt{n}, \iff$$

$$\frac{9}{\epsilon^2} < n.$$

By Archimedian Principle there exists $n_0 \in \mathbb{N}$ such that

$$\frac{9}{\epsilon^2} < n_0.$$

Hence for all $n \geq n_0$ we have

$$\frac{9}{\epsilon^2} < n.$$

Thus for all $n \geq n_0$ we have:

$$|\pi - 3/\sqrt{n} - \pi| < \epsilon.$$

Hence $\lim_{n\to\infty} \pi - 3/\sqrt{n} = \pi$. □