MATHEMATICS 3210. Homework 3: Solutions.

September 13, 2001

1. §3, # 3.1. (a) Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

The answer. A3, A4, M4.

(a) Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{Z}$?

The answer. M4.

§3, # 3.5. (a) Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.

Proof. We have to consider two cases. Case 1: $b \geq 0$. Then the first inequality reads:

$$0 \leq b \leq a.$$

This inequality implies $-a \leq 0 \leq b \leq a$ (since $a \geq 0$ implies $-a \leq 0$), which in turn implies $-a \leq b \leq a$.

On the other hand, if $-a \leq b \leq a$ then $b \leq a$. Thus (in the Case 1) the first inequality implies the second and vice-versa.

Case 2: $b \leq 0$. Then $-b \geq 0$ and applying the analysis from the Case 1 we conclude that the inequalities $|-b| \leq a$ and $-a \leq -b \leq a$ are equivalent. The inequality $|-b| \leq a$ is equivalent to $|b| \leq a$.

The inequality $-a \leq -b \leq a$ is equivalent to $a \geq b \geq -a$ (by multiplying by -1). Hence, the inequalities $|b| \leq a$ and $-a \leq b \leq a$ are equivalent in this case as well. □

3.5. (b) Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Proof. Consider the case $|a| \geq |b|$ (the other case is obtained by interchanging the roles of a and b). Then the inequality we need to prve reads as

$$|a| - |b| \leq |a - b|,$$

equivalently, $|a| \leq |a - b| + |b|$. But the latter follows from the triangle inequality:

$$|(a - b) + b| \leq |a - b| + |b|. \quad \square$$

§4, # 4.1. I will do only several of these.

Answers: (a) $u = 1, 2, 3$. (e) $1, 2, 3$. (h) NBA. (i) 1, 2, 3. (k) NBA. (r) 1, 2, 3. (u) NBA.

4.12. Prove density of irrational numbers: if $a < b$ then there is an irrational number $x \in \mathbb{R}$ so that $a < x < b$.

Proof. First of all, $\sqrt{2}$ is irrational. Ny the Archimedian pronciple, for each $c \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $\sqrt{2}n > c$. Hence for each $eps > 0$ there exists $n \in \mathbb{N}$ so that $0 < 1/(\sqrt{2}n) < \epsilon$. Note that $y = 1/(\sqrt{2}n)$ is still irrational.

By density of rational numbers, there exists $d \in \mathbb{Q} \cap (a, b)$. We will find an irrational number $x \in (d, b)$ (then $a < x < b$).

Take $b - d = \epsilon$, find irrational number $y \in (0, b - d)$. Then $y + d \in (d, b)$. Since d is rational and y is not, it follows that x is irrational. $\square$

3. Find infimum and supremum of each of the following sets:

a) $E = \{4, 3, 2, 1, 8, 7, 6, 5\}$.

Answer: $\inf(E) = \min(E) = 1$, $\sup(E) = \max(E) = 8$.

b) $E = \{x \in \mathbb{R} | x^2 - 3x - 5 = 0\}$.

Solution. Solving the quadratic equation $x^2 - 3x - 5 = 0$ we find two solutions:

$$x_1 = (3 + \sqrt{9 + 20})/2 = (3 + \sqrt{29})/2$$

$$x_2 = (3 - \sqrt{9 + 20})/2 = (3 - \sqrt{29})/2.$$

Since $\sqrt{29} > 0$ and $-\sqrt{29} < 0$, we have $x_2 < x_1$. This implies that $x_1 = \max(E) = \sup(E)$ and $x_2 = \min(E) = \inf(E)$. $\square$

c) $E = [a, b)$ where $a < b$ are real numbers.

Solution. First of all, since $a = \min(E)$, $a = \inf(E)$ as well. Now, consider the supremum:

Since $E = \{x : a \leq x < b\}$, the number b is an upper bound for E. To see that $b = \sup(E)$ we use the approximation property:

Let $y < b$. If $y \leq a$ then $(y, b] \cap [a, b) = (a, b)$ contains $m = (a+b)/2$ and is therefore nonempty:

$$y \leq a = (a + a)/2 < (a + b)/2 = m, \quad m = (a + b)/2 < (b + b)/2 = b.$$

If $y > a$ then $(y, b] \cap [a, b) = (y, b)$ contains $m = (y + b)/2$ and is therefore nonempty as well:

$$a < y = (y + y)/2 < (y + b)/2 = m, \quad m = (y + b)/2 < (b + b)/2 = b.$$

In both cases the approximation principle implies that $b = sup(E)$.

d) $E = \{p/q \in \mathbb{Q} | p^2 < 2q^2 \text{ and } p, q > 0\}$.

Solution. We first find the infimum of E. Since $p, q > 0$, $p/q > 0$ and thus 0 is a lower bound for E. By the density of rational numbers, each open interval $(0, \epsilon)$ contains a rational number p/q. Any such number will belong to E provided that $\epsilon < \sqrt{2}$. If $\epsilon > \sqrt{2} > 1$ then $(0, \epsilon)$ contains rational number $1 \in E$. Thus by the approximation principle, $0 = \inf(E)$.

I claim that $\sqrt{2} = \sup(E)$. Indeed, if $p^2 < 2q^2$ then $(p/q)^2 < 2$ and thus $p/q < \sqrt{2}$. This implies that $\sqrt{2}$ is an upper bound for E. Let's check that this is the least upper bound. Pick $\epsilon > 0$. If $\epsilon > \sqrt{2}$, then $(\sqrt{2} - \epsilon, \sqrt{2}]$ contains the rational number 1. If $\epsilon < \sqrt{2}$, then

$(\sqrt{2}-\epsilon, \sqrt{2}) \cap \mathbb{Q} \subset E$ and by density property of rational numbers, $(\sqrt{2}-\epsilon, \sqrt{2})$ contains a rational number which is therefore an element of E. Thus the approximation property implies that $\sqrt{2} = \sup(E)$. ☐

(e) $E = \{x \in \mathbb{R} | x = 1 + (-1)^n, \text{ for } n \in \mathbb{N}\}$.

Solution. If $n \in \mathbb{N}$ is even then $x = 1 + (-1)^n$ equals $1 + 1 = 2$. If n is odd then $x = 1 + (-1)^n$ equals $1 - 1 = 0$. Thus $E = \{0, 2\}$. Then 0 is a lower bound for E and 2 is an upper bound for E. Since $0 \in E$ and $2 \in E$, the approximation principle implies that $0 = \inf(E)$, $2 = \sup(E)$. ☐

(f) $E = \{x \in \mathbb{R} | x = 1/n - (-1)^n, \text{ for } n \in \mathbb{N}\}$.

Solution. If n is even then $x = 1/n - (-1)^n = 1/n - 1$. If n is odd then $x = 1/n - (-1)^n = 1/n + 1$. Note that

$$1/n - 1 \leq 1 + 1/n, \text{ for all } n \in \mathbb{N}.$$

On the other hand, $1 + 1/n \leq 1 + 1/1 = 2$, for all $n \in \mathbb{N}$. Thus 2 is an upper bound for E and since $2 \in E$ (note that 1 is an odd number), the approximation principle implies that $2 = \sup(E)$. We now find $\inf(E)$. Note that $1 + 1/n \geq 1/n - 1 > -1$ for all n. Thus -1 is a lower bound for E. To check that $-1 = \inf(E)$ we again use the approximation property. Let $\epsilon > 0$. By the Archimedian principle,

$$2k > 1/\epsilon$$

for some $k \in \mathbb{N}$. Thus $\epsilon > \frac{1}{2k}$ and

$$-1 < -1 + \frac{1}{2k} < -1 + \epsilon$$

and $-1 + \frac{1}{2k} \in [-1, -1 + \epsilon)$. Since $-1 + \frac{1}{2k} \in E$, the intersection $E \cap [-1, -1 + \epsilon)$ is nonempty. Therefore by the approximation property, $-1 = \inf(E)$. ☐

(g) $E = \{1 + (-1)^n/n : n \in \mathbb{N}\}$.

Solution. If n is even then $x = 1 + (-1)^n/n = 1 + 1/n$. If n is odd then $x = 1 + (-1)^n/n = 1 - 1/n$. Note that

$$1 - 1/n < 1 < 1 + 1/n$$

for all n. If $n \in \mathbb{N}$ is even then $n \geq 2$, thus $1 + 1/n \leq 1 + 1/2 = 1.5$. Thus 1.5 is an upper bound for E. It follows from the approximation property that $1.5 = \sup(E)$ (since $1.5 \in E$).

If $n \in \mathbb{N}$ is odd then $n \geq 1$, thus $1 - 1/n \geq 1 - 1 = 0$. Hence $0 \in E$ is a lower bound for E and the approximation principle implies that $0 = \inf(E)$. ☐