MATHEMATICS 3210-1. Homework 12: Solutions.

December 7, 2001

1. Section 33, # 33.4, 33.5, 33.7, 33.8, 33.13.

33.4. Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.

Solution. Take $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \notin \mathbb{Q}$. This function is similar to the Dirichlet's function, so we have:

$$U(f) = 1/2, L(f) = -1/2.$$

Hence f is not integrable. On the other hand, $|f(x)| = x$ which is an integrable function. □

33.5. Show that $|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)dx| \leq \frac{16\pi^3}{3}$.

Solution. Note that $|\sin^8(e^x)| \leq 1$, hence

$$|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)dx| \leq \int_{-2\pi}^{2\pi} |x^2||\sin^8(e^x)|dx \leq \int_{-2\pi}^{2\pi} x^2 dx.$$

Since $\int x^2 dx = \frac{x^3}{3} + const$, we have:

$$\int_{-2\pi}^{2\pi} x^2 dx = \frac{1}{3}[(2\pi)^3 - (-2\pi)^3] = \frac{2}{3}8\pi^3 = \frac{16\pi^3}{3}. \quad \square$$

33.7. Let f be a function on $[a,b]$ so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a,b]$.

(a) Show that $U(f^2, P) - L(f^2, P) \leq 2B[U(f,P) - L(f,P)]$ for all partitions P of $[a,b]$.

Solution. For each interval $[t_{k-1}, t_k]$ of the partition P we have:

$$M(f^2, [t_{k-1}, t_k]) = M(f, [t_{k-1}, t_k])^2, m(f^2, [t_{k-1}, t_k]) = m(f, [t_{k-1}, t_k])^2$$

$$U(f^2, P) - L(f^2, P) = \sum_{k=1}^{n} [M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])](t_k - t_{k-1}) =$$

$$= \sum_{k=1}^{n} (t_k - t_{k-1})[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])] \cdot [M(f, [t_{k-1}, t_k]) + m(f, [t_{k-1}, t_k])] \leq$$

$$\sum_{k=1}^{n} (t_k - t_{k-1}) 2B[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])] = 2B[U(f,P) - L(f,P)]. \quad \square$$

(b) Show that if f is integrable on $[a,b]$ then f^2 is also integrable.

Solution. Since f is integrable, for each $\epsilon > 0$ there is $\delta > 0$ so that if $mesh(P) < \delta$ then $U(f,P) - L(f,P) < \frac{\epsilon}{2B}$. Thus (according to (a))

$$U(f^2,P) - L(f^2,P) \le 2B[U(f,P) - L(f,P)] \le 2B\frac{\epsilon}{2B} = \epsilon.$$

This implies that f^2 is also integrable. □

33.8. Let f, g be integrable on $[a,b]$.

(a) Show that fg is integrable on $[a,b]$.

Solution. We know that $f+g$ is integrable, hence $(f+g)^2$ is integrable too (see previous problem). The functions f^2, g^2 are integrable as well. Thus the function

$$\frac{1}{2}[(f+g)^2 - f^2 - g^2]$$

is also integrable. On the other hand,

$$(f+g)^2 = f^2 + 2fg + g^2, fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$$

Thus fg is integrable. □

33.13. Suppose thatf, g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists $x \in [a,b]$ such that $f(x) = g(x)$.

Solution. Consider the function $h = f - g$. This function is continuous on $[a,b]$, $\int_a^b h = \int_a^b f - \int_a^b g = 0$. Hence by the mean value theorem for integrabls, there exists $x_0 \in [a,b]$ such that $h(x)(b-a) = \int_a^b h = 0$. Thus $h(x) = 0$. Hence $f(x) = g(x)$. □

2. Section 34, # 34.2, 34.3, 34.11.

34.2. (a) Calculate

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}dt.$$

Solution. Let $F(x) = \int_0^x e^{t^2}dt$. Then $F'(x) = e^{x^2} = f(x)$ for each x. Note that $F(0) = 0$. Hence

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}dt = \lim_{x\to 0}\frac{1}{x}(F(x) - F(0)) = F'(0) = f(0) = 1.$$

(b) Similar.

34.3. Let f be defined as follows: $f(t) = 0$ for $t < 0$ and $f(t) = t$ for $0 \le t \le 1$, $f(t) = 4$ for $t > 0$.

(a) Compute the function $F(x) = \int_0^x f(t)dt$.

Solution. Case 1: $x \le 0$. Then $F(x) = \int_0^x f(t)dt = \int_x^0 f(t) = 0$.

Case 2. $0 < x \le 1$. Then $F(x) = \int_0^x f(t)dt = \int_0^x tdt = x^2/2$.

Case 3. $1 < x < \infty$. Then

$$F(x) = \int_0^1 f(t)dt + \int_1^x f(t)dt = 0.5 + \int_1^x 4dt = 0.5 + 4(x-1) = 4x - 3.5.$$

(b) Is f continuous?

Solution. F is continuous since it is integral of a bounded function (theorem 34.3).

(c) Where is F differentiable? Compute the derivative.

Solution. By inspection, F is differentiable for each $x \neq 0, 4$. The derivative of F on the intervals $(-\infty, 0), (0, 1), (1, \infty)$ is $0, t, 4$ respectively. Let's verify that F is not differentiable at 0 (similar argument works for $x = 1$). Consider

$$\lim_{x\to 0-} \frac{F(x) - F(0)}{x} = \lim_{x\to 0-} 0 = 0,$$

$$\lim_{x\to 0+} \frac{F(x) - F(0)}{x} = \lim_{x\to 0+} 1 = 1.$$

Thus $\lim_{x\to 0} \frac{F(x)-F(0)}{x}$ does not exist, so the function F is not differentiable at zero. The same works at $x = 1$. □

34.11. Suppose that f is a continuous function on $[a, b]$ and that $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Consider the function $F(x) = \int_a^x f(t)dt$. For $x < y$ we have

$$F(y) - F(x) = \int_x^y f(t)dt \geq 0.$$

Hence F is increasing. On the other hand, $F(a) = 0, F(b) = \int_a^b f(t)dt = 0$. Thus $F(x) = 0$ for each $x \in [a, b]$. Hence, $f(x) = F'(x) = 0$ for each $x \in [a, b]$. □