MATHEMATICS 3210-1. Homework 11: Solutions.

December 3, 2001

1. 1. Compute (directly, using the definition) the Taylor series about the origin for the function $f(x) = \frac{1}{1-x}$. Show that the Taylor series converges to $f(x)$ on the interval $(-1, 1)$. Now do the computation using theorem 31.7.

Solution. First, consider the derivatives of the function $f(x)$:

$f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$,..., $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$. Let's verify the latter formula using the induction. The assertion is true for $n = 1$ (the first derivative). Suppose that

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}.$$

Then

$$f^{(k+1)}(x) = (f^{(k)})'(x) = (\frac{k!}{(1-x)^{k+1}})' = \frac{k!(k+1)}{(1-x)^{k+2}} = \frac{(k+1)!}{(1-x)^{k+2}}.$$

Thus $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ for all n.

Thus $f^{(n)}(0) = n!$. Hence the Taylor series for f equals

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} x^k.$$

To prove that the Taylor series converges to f we need to estimate the higher derivatives of the function f.

Suppose that $x \in (-1, 1), x \neq 0$. I will consider the case $x > 0$ since the case $x < 0$ is similar.

By integral for of Taylor's theorem (theorem 31.5), for each n we have:

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt = n \int_0^x \frac{(x-t)^{n-1}}{(1-t)^{n+1}} dt.$$

Note that $\frac{x-t}{1-t} < x$, hence

$$0 < n \int_0^x \frac{(x-t)^{n-1}}{(1-t)^{n+1}} dt \leq n \int_0^x \frac{x^{n-1}}{(1-t)^2} dt = \frac{nx^n}{1-x}.$$

Thus we would like to show that

$$\lim_{n \to \infty} nx^n = 0.$$

We will prove that there is $n_0 \in \mathbb{N}$ such that the sequence nx^n is decreasing for all $n \geq n_0$. Indeed,

$$(n+1)x^{n+1} < nx^n \iff (n+1)x < n \iff x < \frac{n}{n+1}.$$

Note that the sequence $\frac{n}{n+1}$ converges to 1, and $0 < x < 1$; hence there exists n_0 so that for all $n \geq n_0$ we have

$$x < \frac{n}{n+1}.$$

Thus the sequence (nx^n) is indeed decreasing for all $n \geq n_0$. Since this sequence is bounded from below by zero, it converges to some number $a \geq 0$. Thus

$$a = \lim_{n\to\infty}(n+1)x^{n+1} = \lim_{n\to\infty} nx^n x + x^{n+1} = ax,$$

since the sequence x^{n+1} converges to zero. Thus $a = ax$, hence either $a = 0$ or $x = 1$. Since $x < 1$ we conclude that $a = 0$. Thus the sequence $\frac{nx^n}{1-x}$ converges to zero as well, by sandwich lemma we conclude that the sequence $R_n(x)$ converges to zero too. Hence

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 ...$$

□

2. Section 31, # 31.2. Find the Taylor series for $\cos(x)$ and indicate why it converges to $\cos(x)$ for $x \in \mathbb{R}$.

Solution. We solved this in the class.

31.5. Let $g(x) = e^{-1/x^2}$ for $x \neq 0$ and $g(0) = 0$. Show that $g^{(n)}(0) = 0$ for all n. Show that the Taylor series for g agrees with g only at $x = 0$.

Solution. We argue similarly to the Example 3. For $x \neq 0$ we have $g'(x) = \frac{2}{x^3}g(x)$. I claim that n-th order derivative of g at $x \neq 0$ has the form $r_n(x)g(x)$ where $r_n(x)$ is a rational function of x. I will prove it using the induction. The assertion is true for $n = 1$ since $\frac{2}{x^3}$ is a rational function. Suppose that $g^{(k)}(x) = r_k(x)g)(x)$ where $r_k(x)$ is a rational function of x. Then

$$g^{(k+1)}(x) = (r_k(x)g)(x))' = r_k'(x)g(x)+r_k(x)g'(x) = r_k'(x)g(x)+\frac{2}{x^3}g(x) = (r_k'(x)+\frac{2}{x^3})g(x).$$

Since the derivative of a rational function is again rational and the sum of two rational functions is also rational, we conclude that $r_{k+1}(x) = r_k'(x) + \frac{2}{x^3}$ is a rational function. Thus $g^{(k+1)}(x) = r_{k+1}(x)g(x)$ where $r_{k+1}(x)$ is a rational function. Thus the claim is proven.

Now I will prove that $g^{(n)}(0) = 0$ again using induction. For $n = 1$ we get:

$$g'(0) = \lim_{x\to 0}\frac{g(x)}{x} = \lim_{x\to 0}\frac{e^{-1/x^2}}{x}.$$

I first consider the limit from the right. Note that for $0 < x < 1$ we have $x^2 < x$ and thus $\frac{-1}{x^2} < \frac{1}{x}$. Hence

$$\frac{e^{-1/x^2}}{x} < \frac{e^{-1/x}}{x}$$

for $x \in (0,1)$; $\lim_{x\to 0} \frac{e^{-1/x}}{x} = 0$. Thus by sandwich lemma, $\lim_{x\to 0+} \frac{e^{-1/x^2}}{x} = 0$.

To compute the limit from the left note that if $x < 0$ then $y = -x > 0$. Then

$$\lim_{x\to 0-} \frac{e^{-1/x^2}}{x} = \lim_{y\to 0+} \frac{e^{-1/y^2}}{-y} = -\lim_{y\to 0+} \frac{e^{-1/y^2}}{y} = 0.$$

Therefore $\lim_{x\to 0} \frac{e^{-1/x^2}}{x} = 0$ and hence $g'(0) = 0$.

Now suppose that $f^{(k)}(0) = 0$. Then

$$g^{(k+1)}(0) = \lim_{x\to 0} \frac{r_k(x) g(x)}{x},$$

since $g^{(k)}(x) = r_k(x)g(x)$ for $x \neq 0$. Again, consider first the limit from the right. Then for $x > 0$ we have $g(x) < e^{-1/x}$. Hence it suffices to show that

$$\lim_{x\to 0+} \frac{r_k(x)e^{-1/x}}{x} = 0.$$

The ratio $\frac{r_k(x)}{x}$ is a rational function $R(x)$. Let $x = 1/y$. Then, as $x \to 0+$, $y \to \infty$. By substituting x with $1/y$ in the function R we get a rational function

$$\frac{P(y)}{Q(y)} = \frac{a_0 + a_1 y + ... + a_m y^m}{b_0 + b_1 y + ... + b_d y^d}$$

Then

$$\lim_{x\to 0+} R(x)e^{-1/x} = \lim_{y\to\infty} \frac{a_0 + a_1 y + ... + a_m y^m}{e^y(b_0 + b_1 y + ... + b_d y^d)}$$

Since

$$\lim_{y\to\infty} \frac{1}{b_0 + b_1 y + ... + b_d y^d} = 0,$$

$$\lim_{y\to\infty} \frac{a_0 + a_1 y + ... + a_m y^m}{e^y} = 0$$

we conclude that

$$\lim_{x\to 0+} R(x)e^{-1/x} = 0.$$

To get the limit from the left make the substitution $x = -t$, then $R(-t) = F(t)$ is again a rational function of t and

$$\lim_{x\to 0-} R(x)e^{-1/x^2} = \lim_{t\to 0+} R(-t)e^{-1/t^2} == \lim_{t\to 0+} F(t)e^{-1/t^2} = 0.$$

Thus

$$g^{(k+1)}(0) = \lim_{x\to 0} \frac{r_k(x)e^{-1/x}}{x} = 0.$$

In particular, the Taylor series for the function $g(x)$ is zero. The function g is positive for all $x \neq 0$, hence $g(x)$ agrees with the Taylor series only for $x = 0$. □

3. Section 32, # 32.1, 32.2, 32.8.

32.1 Find the upper and lower Darboux integrals for $f(x) = x^3$ on $[0, b]$.
Solution. We first prove a couple of claims.
Claim. $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$.

Proof. Let's verify this using the induction on n. For $n = 1$ we get $1 = 1$ which is true.

Suppose that $\sum_{i=1}^{k} i^3 = (\sum_{i=1}^{k} i)^2$. We want to prove that

$$\sum_{i=1}^{k+1} i^3 = (\sum_{i=1}^{k+1} i)^2.$$

Equivalently,

$$(\sum_{i=1}^{k} i^3) + (k+1)^3 = ((\sum_{i=1}^{k+1} i) + (k+1))^2 = (\sum_{i=1}^{k+1} i)^2 + 2(\sum_{i=1}^{k+1} i)(k+1) + (k+1)^2.$$

By using the induction hypothesis, the latter is equivalane to

$$(\sum_{i=1}^{k+1} i)^2 + (k+1)^3 = (\sum_{i=1}^{k+1} i)^2 + 2(\sum_{i=1}^{k+1} i)(k+1) + (k+1)^2. \iff$$

$$(k+1)^3 = (k+1)^2 + 2(k+1)(1+2+...+k) \iff$$

$$(k+1)^2 = (k+1) + 2(1+2+...+k) \iff (k+1)^2 - (k+1) = 2(1+2+...+k) \iff$$

$$\frac{k(k+1)}{2} = 1+2+...+k.$$

The last is a true equality, hence we are done by the induction. □

Corollary.

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}. \quad \square$$

We now start computing the upper and lower Darboux integrals.

Let P be a partition $t_0 = 0 < t_1 < ... < t_n = b$. Then $\sup(f|_{[t_{i-1},t_i]}) = f(t_i) = t_i^3$ and $\inf(f|_{[t_{i-1},t_i]}) = f(t_{i-1}) = t_{i-1}^3$. We consider only the partitions P_n of $[0,b]$ into equal intervals, i.e.,

$$t_i = ib/n.$$

We will prove that $\lim U(f, P_n) = \lim L(f, P_n) = U(f) = L(f)$, thus by theorem 32.5, $U(f) = L(f)$. Hence it suffices to consider only the partitions of the above form. Then

$$U(f, P_n) = \sum_{i=1}^{n} t_i^3(t_i - t_{i-1}) = \sum_{i=1}^{n} i^3 b^4/n^4 = \frac{b^4}{n^4}\sum_{i=1}^{n} i^3 = \frac{b^4}{n^4}(1+2+...+n)^2 = \frac{b^4}{n^4}\frac{(n(n+1))^2}{4}.$$

Similarly,

$$L(f, P_n) = \sum_{i=1}^{n} t_{i-1}^3(t_i - t_{i-1}) = \sum_{i=1}^{n}(i-1)^3 b^4/n^4.$$

Since $\sum_{i=1}^{n}(i-1)^3 = (\sum_{i=1}^{n} i^3) - n^3$, we get

$$L(f, P_n) = \frac{b^4}{n^4}\frac{(n(n+1))^2}{4} - \frac{b^4 n^3}{n^4} = \frac{b^4}{4}\frac{(n(n+1))^2}{n^4} - b^4/n = U(f, P_n) - b^4/n.$$

Note that

$$\lim_{n\to\infty} b^4/n = 0, \lim_{n\to\infty} \frac{(n(n+1))^2}{n^4} = 1.$$

Thus

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \frac{b^4}{4}.$$

Thus $U(f) = L(f) = \frac{b^4}{4}$. □

32.2. Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Compute the upper and lower Darboux integrals for f on $[0, b]$.

(b) Is f integrable on $[0, b]$?

Solution. Let $P : t_0 = 0 < t_1 < ... < t_n = b$ be a partition of $[0, b]$. Then the upper Darboux sum for this partition is

$$U(f, P) = \sum_{i=1} n \sup(f|_{[t_{i-1}, t_i]})(t_i - t_{i-1}).$$

Since each interval $[t_{i-1}, t_i]$ contains a sequence of rational numbers which converges to t_i, the supremum of the function f on $[t_{i-1}, t_i]$ is $f(t_i)$. Thus

$$U(f, P) = \sum_{i=1} nt_i(t_i - t_{i-1}).$$

Similarly, since each interval $[t_{i-1}, t_i]$ contains an irrational number, the infimum of the function f on $[t_{i-1}, t_i]$ is zero. Thus

$$L(f, P) = 0.$$

Among all the partitions $P : t_0 = 0 < t_1 < ... < t_n = b$ the partition with the maximal sum

$$\sum_{i=1} nt_i(t_i - t_{i-1})$$

is the partition with equal distances $t_i - t_{i-1} = \frac{b}{n}$. For such partition

$$U(f, P) = \sum_{i=1} nt_i \frac{b}{n} = \sum_{i=1} n \frac{bi}{n}\frac{b}{n} = \frac{b^2}{n^2} \sum_{i=1} ni = \frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2}{2}\frac{n+1}{n}.$$

The sequence $\frac{n+1}{n}$ is decreasing and it converges to 1, hence

$$U(f) = \inf_n U(f, P) = \frac{b^2}{2}.$$

Thus $U(f) = b^2/2$. In particular, $U(f) \neq L(f)$ and hence f is not integrable on $[0, b]$. □

32.8. Show that if f is integrable on $[a, b]$ then f is integrable on every interval $[c, d] \subset [a, b]$.

Proof. I first consider the case $d = b$, $a < c < b$. Recall that f if integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ so that for each partition T of $[a, b]$ with $mesh(T) < \delta$, we have

$$U(f, T) - L(f, T) < \epsilon.$$

Let P, Q be partitions of $[a, c], [c, b]$ of the mesh $< \delta$. Then $T = P \cup Q$ is a partition of $[a, b]$ of mesh $< \delta$. Then

$$U(f, T) - L(f, T) = U(f, P) - L(f, P) + U(f, Q) - L(f, Q) < \epsilon.$$

Since both numbers $U(f, P) - L(f, P)$ and $U(f, Q) - L(f, Q)$ are positive, we conclude that

$$U(f, P) - L(f, P) < \epsilon, U(f, Q) - L(f, Q) < \epsilon.$$

Hence f is integrable on both $[a, c]$ and $[c, b]$. To get the general case note that integrability of f on $[a, b]$ implies integrability of f on $[a, d]$ (by what we just proved). In turn, integrability of f on $[a, d]$ implies integrability of f on $[c, d]$. □