MATHEMATICS 3210-1. Homework 10: Solutions.

November 15, 2001

1. Section 28, # 28.8. Let $f(x) = x^2$ if x is rational and $f(x) = 0$ for irrational x.
(a) Prove that f is continuous at 0.
(b) prove that f is discontinuous at $x \neq 0$.
(c) Prove that f is differentiable at $x = 0$.
Solution. (a) will follow from (c).
(b) Let x be irrational (thus $x \neq 0$). Then there exists a sequence of rational numbers x_n such that $\lim x_n = x$. Then $f(x_n) = x^2$, $\lim f(x_n) = x^2 \neq 0$. However $f(x) = 0$ since x is irrational. Thus f is discontinuous at x. The same argument applies to rational $x \neq 0$.
(c) Consider

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

I claim that this limit equals 0. This will show that $f'(0)$ exists and equals zero. To prove this consider a sequence $x_n \to 0, x_n \neq 0$. Then $|f(x_n)| \leq x_n^2$ (the equality holds for $x_n \in \mathbb{Q}$ and the inequality $0 < x_n^2$ holds for irrational x_n). Thus,

$$|f(x_n)/x_n| \leq |x_n^2/x_n| = |x_n|$$

$\lim |x_n| = 0$. Hence by the sandwich lemma we have:

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{|f(x)|}{|x|} = 0. \quad \square$$

28.12. Differentiate the function $f(x) = \cos(e^{x^5-3x})$.
Solution. Let $g(x) = x^5 - 3x$, $h(y) = e^y = \exp(y)$, $k(z) = \cos(z)$. Then $f = k \circ h \circ g$. Hence by the chain rule we have:

$$f'(x) = \cos'(e^{x^5-3x}) \cdot \exp'(x^5 - 3x) \cdot (5x^4 - 3) = -\sin(e^{x^5-3x}) e^{x^5-3x}(5x^4 - 3). \quad \square$$

28.14. Suppose that f is differentiable at a. Prove that
(a) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$.

Proof. Consider the functions $x = g(h) = h + a$ (this function is continuous), $k(x) = \frac{f(x)-f(a)}{x-a}$. Then $\lim_{h\to 0} g(h) = a$, thus by theorem 20.5,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} k \circ g(h) = \lim_{x\to a} k(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a). \quad \square$$

(b) $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

Proof. Let $k = g(h) = -h$. Then by theorem 20.5 and part (a) we have:

$$f'(a) = \lim_{k\to 0} \frac{f(a+k) - f(a)}{k} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}.$$

Thus

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2}[\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h}],$$

hence

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2}[\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}] =$$

$$= \frac{1}{2}[f'(a) + f'(a)] = f'(a). \quad \square$$

2. Section 29, # 29.2. Prove that $|\cos(x) - \cos(y)| \leq |x - y|$.

Proof. Recall that $\cos'(t) = -\sin(t)$, hence $|\cos(t)| \leq |\sin(t)| \leq 1$. Thus by the mean value theorem we have

$$\cos(x) - \cos(y) = \cos'(t)(x - y)$$

for some $t \in [x, y]$, hence

$$|\cos(x) - \cos(y)| = |\cos'(t)| \cdot |x - y| \leq |x - y|. \quad \square$$

29.5. Suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

Proof.

$$|\frac{f(x) - f(y)}{x - y}| \leq |x - y|.$$

Hence, by the sandwich lemma, for each $a \in \mathbb{R}$ we have

$$\lim_{x\to a} |\frac{f(x) - f(a)}{x - a}| = \lim_{x\to a} |x - a| = 0.$$

Thus $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0$, hence $f'(a) = 0$ for each $a \in \mathbb{R}$. Thus f is constant (Corollary 29.4). $\square$

29.9. Show that $ex \leq e^x$ for all $x \in \mathbb{R}$.

Solution. First consider $1 \leq x < \infty$. For $x = 1$ we have $e = e^1$, so the equality holds. Now, for each $x > 1$ we have:

$$(ex)' = e < e^x$$

since e^x is strictly increasing. Hence for each $x \in [1, \infty)$ we have

$$ex \leq e^x$$

with equality only if $x = 1$.

Now, consider $-\infty < x \le 1$ and define $f(x) = ex - e^x$. Suppose that there exists $x \in (-\infty, 1)$ such that $f(x) > 0$, i.e. $ex > e^x$. Then by the minvalue theorem, there exists $c \in (x, 1)$ such that

$$e - e^c = f'(c) = \frac{f(x) - f(1)}{x - 1} = \frac{f(x)}{x - 1}.$$

Note that for $c < 1$ we have $e > e^c$, i.e. $e - e^c > 0$. On the other hand, $f(x) > 0$ but $x - 1 < 0$ (recall that $x < 1$). Thus in the equality

$$e - e^c = \frac{f(x)}{x - 1}$$

the left hand side is positive but the right hand side is negative. Contradiction. Hence $ex \le e^x$ for each $x \le 1$. □

29.11. Show that $\sin(x) \le x$ for all $x \ge 0$.

Proof. Consider the function $f(x) = x - \sin(x)$ on $[0, \infty)$. Then $f(0) = 0$, for each $x > 0$ we have

$$f'(x) = 1 - \cos(x) \ge 0.$$

Hence f is increasing, so $f(x) \ge f(0) = 0$. Thus $x \ge \sin(x)$ for each $x \ge 0$. □

29.16. Obtain the derivative for the function $\arctan(x)$, for $x \in \mathbb{R}$.
Solution. First note that

$$\tan'(y) = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}.$$

$\arctan(x)$ is the inverse function to $f(y) = \tan(y)$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, if $x = \tan(y) = \frac{\sin(y)}{\cos(y)}$,

$$x^2 + 1 = \frac{\sin(y)^2}{\cos^2(y)} + 1 = \frac{1}{\cos^2(y)}.$$

Hence

$$\frac{1}{1 + x^2} = cos^2(y).$$

On the other hand, by the chain rule we have

$$\arctan'(x) = \frac{1}{\tan'(y)} = \cos^2(y).$$

Thus $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$. □