MATHEMATICS 3210. Solutions for Homework # 1.

1. Write the negation of each statement:
(a) M is a symmetric matrix.
(b) Today is Wednesday or it is snowing.
(c) Bob and Betty are related.
(d) If it rains today, then the roof will leak.
(e) If roses are red and violets are blue, then I love you.

Solution. (a) First, let's expand the statement "M is a symmetric matrix":
"The number of rows in M is the same as the number of columns (A)
AND
for each pair of induces i, j we have $m_{ij} = m_{ji}$ (B)".
(Note the universal quantifier in (B)!)
Thus the negation of the above statement has the form "NOT (A) OR NOT B".

The negation of (A) is "The number of columns is different from the number of rows", the negation of (B) is "There exists a pair of indices (i, j) such that $m_{ij} \neq m_{ji}$". Hence the negation of the entire statement is:

"The number of columns is different from the number of rows AND there exists a pair of indices (i, j) such that $m_{ij} \neq m_{ji}$".

(b) "Today is not Wednesday AND it is now snowing".

(c) This is a trick question, note that AND in the above sentance does not connect statements, it connects names. Hence the negation is

"Bob and Betty are not related " (equivalently, "Bob and Betty are unrelated ").

(d) The statement has the form of the implication $A \Rightarrow B$. The only case when the implication is false is when the assumption is true but the conclusion is false. Hence the negation is

"It rains today AND the roof does not leak" (equivalently, "It rains today but the roof does not leak").

(e) The statement is again implication $(A\&B) \Rightarrow C$, where A: "The roses are red", B: "The violets are blue", C:"I love you".

Hence the negation is "$(A\&B)$ AND NOT C". Hence the negation of the entire statement is

"The roses are red and violets are blue AND I do not love you."

2. Show that for any sets B, A_1, A_2 we have $B \backslash (A_1 \cup A_2) = (B \backslash A_1) \cap (B \backslash A_2)$.

Solution. a) We first show that each $x \in B \backslash (A_1 \cup A_2)$ belongs to $(B \backslash A_1) \cap (B \backslash A_2)$.

$x \in B$ does not belong to $A_1 \cup A_2$, hence it belongs to neither A_1 nor to A_2. Hence $x \in B \backslash A_1$ and $x \in B \backslash A_2$ which means that $x \in (B \backslash A_1) \cap (B \backslash A_2)$.

b) We now show that each $x \in (B \backslash A_1) \cap (B \backslash A_2)$ belongs to $B \backslash (A_1 \cup A_2)$.

The element x belongs to both $B \backslash A_1$ and to $B \backslash A_2$. Thus $x \in B$ and at the same time, x belongs to neither A_1, nor A_2. Thus $x \in B$ does not belong to the union $A_1 \cup A_2$. This means that $x \in B \backslash (A_1 \cup A_2)$. □

3. Problem # 1 from the handout. For $f : A \to B$, compute $f(A_0), f^{-1}(B_0)$.
(a) $f(x) = x^2$, $f : A = \{-1, 0, 1\} \to B = \mathbb{R}$. $A_0 = \{-1, 1\}$, $B_0 = \{0, 1\}$.

Solution. $f(A_0) = \{f(-1), f(1)\}$. Since $f(-1) = (-1)^2 = 1, f(1) = 1^1 = 1$ we get:

$$f(A_0) = \{f(-1), f(1)\} = \{1, 1\} = \{1\}.$$

To compute $f^{-1}(B_0)$ note that this set equals $\{x \in A : f(x) \in B_0\} = \{x \in A : f(x) = 0\} \cup \{x \in A : f(x) = 1\}$. We have:

$$\{x \in A : f(x) = 0\} = \{x \in \{-1, 0, 1\} | x^2 = 0\} = \{0\}.$$

$$\{x \in B : f(x) = 1\} = \{x \in \{-1, 0, 1\} | x^2 = 1\} = \{1, -1\}.$$

Hence $f^{-1}(B_0) = \{0\} \cup \{1, -1\} = \{-1, 0, 1\}$. □

(b) $A = B = \mathbb{R}$, $A_0 = \{x \in \mathbb{R} | x > 0\}$, $B_0 = \{0\}$,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Solution. To compute $f(A_0)$ note that $A_0 \subset \{x | x \geq 0\}$, hence the restriction of f to A_0 is given by $f(x) = x^2$. Square of each $x > 0$ is a positive real number. On the other hand, each positive real number y is the square of another positive real number, namely, $\sqrt{y}$. Hence $f(A_0) = \{x \in \mathbb{R} | x > 0\}$.

To find $f^{-1}(B_0)$ note that this set equals $\{x \in \mathbb{R} : x^2 = 0\} = \{0\}$. □

(c) $A = B = \mathbb{R}$, $A_0 = B_0 = (-2, 1) = \{x \in \mathbb{R} | -2 < x < 1\}$,

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Solution. The function f is given by three different formulae on three different subsets of the domain. To compute $f(A_0)$ we break the set A_0 into the subsets: $A_0 = (-2, 0) \cup \{0\} \cup (0, 1)$ according to the definition of the function f so that on each of these subsets the function f is given by single formula. Then $f(A) = f((-2, 0)) \cup f(\{0\}) \cup f((0, 1))$.

$$f((-2, 0)) = \{-1\}, f(\{0\}) = \{0\}, f((0, 1)) = \{1\}.$$

Hence $f(A) = \{-1\} \cup \{0\} \cup \{1\} = \{-1, 0, 1\}$.

To find $f^{-1}(B_0)$ we again break the domain into three subsets and

$$f^{-1}(B_0) = (f^{-1}(B_0) \cap (-\infty, 0)) \cup (f^{-1}(B_0) \cap \{0\}) \cup (f^{-1}(B_0) \cap (0, -\infty)).$$

We now compute each of these three subsets. (i)

$$f^{-1}(B_0) \cap (-\infty, 0) = \{x | x < 0 \text{ and } f(x) = -1 \in B_0\}.$$

Since $-1 \in B_0$ we get: $f^{-1}(B_0) \cap (-\infty, 0) = (-\infty, 0)$.

(ii)

$$f^{-1}(B_0) \cap \{0\} = \{x \in \mathbb{R} | x = 0, f(x) \in B_0\} = \{0\}$$

since $f(0) = 0 \in B_0$.

(iii)

$$f^{-1}(B_0) \cap (0, \infty) = \{x | x > 0 \text{ and } f(x) = 1 \in B_0\} = \emptyset,$$

since $f(x) = 1 \notin B_0$ for each $x > 0$.

Therefore

$$f^{-1}(B_0) = (-\infty, 0) \cup \{0\} \cup \emptyset = (-\infty, 0]. \quad \square$$