Solutions for the sample of the 2-nd midterm test.

1. Does the following limit exists? Explain your solution.

$$\lim_{(x,y)\to(0,0)} \frac{2x-y}{x+3y}.$$

Solution. The limit does not exist. To verify this claim consider the limit along the x-axis:

$$\lim_{(x,0)\to(0,0)} \frac{2x-0}{x+0} = 2.$$

Now consider the limit along the y-axis:

$$\lim_{(0,y)\to(0,0)} \frac{0-y}{0+3y} = -1/3.$$

The limits are different, hence the limit of the function of two variables does not exist.

2. Find the Cartesian equation corresponding to the following spherical coordinates equation:

$$(\rho\sin(\phi))^2 = \rho\cos(\phi).$$

Solution. $\rho\cos(\phi) = z$, $\rho\sin(\phi) = \sqrt{x^2+y^2}$, hence the equation becomes:

$$x^2 + y^2 = z.$$

3. Find equation (in the form $Ax + By + Cz + D = 0$) of the tangent plane Π to the surface $x^2 + y^2 - z^2 = 1$ at the point $P_0 = (1, 1, 1)$.

Solution. $\nabla f(x, y, z) = (2x, 2y, -2z) = (2, 2, -2)$. Thus we can use the vector $\overrightarrow{n} = (1, 1, -1)$ as the normal vector. Therefore the equation of the plane is

$$(x, y, z)\cdot\overrightarrow{n} = (1, 1, 1)\cdot\overrightarrow{n},$$

$$x + y - z = 1 + 1 - 1 = 1.$$

Thus the answer is: $x + y - z - 1 = 0$.

4. Find the critical points of the function

$$f(x, y) = 2x^2 - \frac{1}{3}y^3 + xy^2 - 3x$$

and determine which of them are local minima, maxima, saddle points.

Solution. The function is differentiable on the whole plane, there are no boundary points, hence the only critical points are the stationary points.

$\nabla f(x, y) = (4x + y^2 - 3, -y^2 + 2xy)$. The stationary points satisfy:

$$\nabla f(x, y) = (0, 0); \quad 4x + y^2 - 3 = 0, -y^2 + 2xy = 0.$$

From the 2-nd equation we get: either $y = 0$ or $y = 2x$. If $y = 0$ then $x = 3/4$ and hence $P_0 = (3/4, 0)$ is the first stationary point. If $y = 2x$ then $2y + y^2 - 3 = 0$ which has two solutions: $y_1 = 1, y_2 = -3$. If $y = y_1 = 1$ then $x = x_1 = 1/2$. If $y = y_2 = -3$

then $x = x_2 = -3/2$. Thus we get three critical points: $P_0 = (3/4, 0)$, $P_1 = (1/2, 1)$, $P_2 = (-3/2, -3)$. The determinant of the 2-nd derivatives is

$$D(P) = \begin{vmatrix} 4 & 2y \\ 2y & -2y + 2x \end{vmatrix}.$$

If $P = P_0$ we get:

$$D(3/4, 0) = \begin{vmatrix} 4 & 0 \\ 0 & 3/2 \end{vmatrix} = 6 > 0$$

Since $f_{xx}(P_0) = 4 > 0$, the point P_0 is the point of local minimum.

If $P = P_1$ we get:

$$D(1/2, 1) = \begin{vmatrix} 4 & 2 \\ 2 & -2 + 1 \end{vmatrix} = -4 - 4 = -8 < 0$$

hence P_1 is a saddle point.

If $P = P_2$ we get:

$$D(-3/2, -3) = \begin{vmatrix} 4 & -6 \\ -6 & 6 - 3 \end{vmatrix} = 12 - 36 = -24 < 0$$

hence P_2 is a saddle point.

5. Use Lagrange's method to find the minimum point(s) of the function $f(x, y) = (x-1)^2 + (y-1)^2$ subject to the constrain $(x+1)(y+1) = 0$. You can assume that the minimum exists.

Solution. Let $g(x, y) = (x+1)(y+1)$, then $\nabla g(x, y) = (y+1, x+1)$, $\nabla f(x, y) = 2(x-1, y-1)$. Thus we get:

$$(x-1, y-1) = \lambda(y+1, x+1), \quad (x+1)(y+1) = 0.$$

If $\lambda \neq 0$ then $x = 1, y = 1$, which contradicts the equation $(x+1)(y+1) = 0$. Hence $\lambda \neq 0$. Then $(x-1)(x+1) = (y-1)(y+1)$, $x^2 = y^2$, $y = \pm x$. Substituting this to the equation $(x+1)(y+1) = 0$ we get: $x = y, (x+1)^2 = 0$, $x = -1 = y$ or $y = -x$, $(x+1)(1-x) = 0$, $x^2 = 1$, $x = \pm 1$, $y = - \pm 1$.

Therefore we got three points where the minimum can occur: $P_0 = (-1, -1), P_1 = (1, -1), P_2 = (-1, 1)$.

Now we compare the values of f at the points where minimum can occur. $f(P_0) = (-1-1)^2 + (-1-1)^2 = 8$, $f(P_1) = (1-1)^2 + (-1-1)^2 = 4$, $f(P_2) = (-1-1)^2 + (1-1)^2 = 4$. Since $4 < 8$ we conclude that the points $P_1 = (1, -1), P_2 = (-1, 1)$ are the points of minimum for the function f on the curve $(x+1)(y+1) = 0$.

Note that the minimum exists since the function f is square of the distance function from the point $(1, 1)$.