

# MATHEMATICS 2210-3. Homework 9: Solution.

March 19, 2001

1. Problems # 4, 12 from Section 15.8. Each problem is worth 10 points.

**Problem 4.** Find all critical points. For each critical point indicate whether it is local minimum, maximum or a saddle point.  $f(x, y) = xy^2 - 6x^2 - 3y^2$ .

**Solution.** The function is defined on the whole plane and is differentiable everywhere, hence its only critical points are the stationary points.

$\nabla f = (y^2 - 12x, 2xy - 6y)$ . If  $\nabla f = \vec{0}$  then

$$y^2 = 12x, 2xy = 6y.$$

One solution is  $x = 0, y = 0$ . If  $y \neq 0$  we can divide the last equation by  $2y$  and get  $x = 3$ , hence  $y^2 = 36, y = \pm 6$ .

Thus we have three critical points:  $P_1 = (0, 0), P_2 = (3, 6), P_3 = (3, -6)$ .

We now consider the 2-nd derivatives:

$$d^2f = \begin{bmatrix} -12 & 2y \\ 2y & -6 \end{bmatrix},$$

the determinant equals  $D(P) = 72 - 4y^2$ .

1. If  $P = P_1$  then  $D(P_1) = 72 > 0$  and  $f_{xx}(P_1) = -12 < 0$ , thus the point  $P_1 = (0, 0)$  is the point of local maximum.

2. If  $P = P_2$  then  $D(P_2) = 72 - 4 \cdot 36 = -72 < 0$ , hence the point  $P_2 = (3, 6)$  is a saddle point.

3. If  $P = P_3$  then  $D(P_3) = 72 - 4 \cdot 36 = -72 < 0$ , hence the point  $P_3 = (3, -6)$  is a saddle point.

**Problem 12.** Find the global maximum value and the global minimum value of  $f$  on  $S$  and indicate where each occurs.

$$f(x, y) = x^2 + y^2, S = \{(x, y) : -1 \leq x \leq 3, -1 \leq y \leq 4\}.$$

**Solution.** Since  $f$  is a continuous function and  $S$  is a closed and bounded set, the function  $f$  attains both global minimum and global maximum.  $\nabla f = (2x, 2y)$  is zero only at  $P_0 = (0, 0)$ . Hence the critical points of  $f$  are  $P_0 = (0, 0)$  and the boundary points of the rectangle  $S$ .

The function  $f$  is the square of the distance to the origin, hence its minimal value is zero and it is attained at the point  $P_0 = (0, 0)$ .

For the same reason, the maximal value of  $f$  is attained at one of the corners of the rectangle  $S$ , we just have to decide which one.

1.  $P_1 = (-1, -1), f(P_1) = 1 + 1 = 2$ .

2.  $P_2 = (-1, 4)$ ,  $f(P_2) = 1 + 16 = 17$ .

3.  $P_3 = (3, -1)$ ,  $f(P_3) = 9 + 1 = 16$ .

4.  $P_4 = (3, 4)$ ,  $f(P_4) = 9 + 16 = 25$ .

Hence 25 is the maximal value of  $f$ , it is attained at the point  $(3, 4)$ .

2. Problems # 2, 10 from Section 15.9.

**Problem 2.** (20 points) Find the maximum of  $f(x, y) = xy$  subject to the constraint  $4x^2 + 9y^2 - 36 = 0$ .

**Solution.** The curve  $4x^2 + 9y^2 - 36 = 0$  is an ellipse, hence it is closed and bounded. Thus the function  $f$  attains its maximum on this curve. Let  $g(x, y) = 4x^2 + 9y^2 - 36$ . The Lagrange equation for the maximum is:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0.$$

Hence

$$(y, x) = \lambda(8x, 18y), \quad 4x^2 + 9y^2 - 36 = 0.$$

Note that if  $x = 0$  then  $y = 0$  (from the 1-st equation), which contradicts the second equation. Thus  $x \neq 0, y \neq 0$  and we get:

$$8x/y = 18y/x, \quad 4x^2 + 9y^2 - 36 = 0.$$

$$4x^2 = 9y^2, \quad 4x^2 + 9y^2 = 36$$

Thus  $8x^2 = 36$ ,  $x^2 = 9/2$ ,  $x = \pm 3/\sqrt{2}$ . Therefore

$$9y^2 = 18, \quad y = \pm\sqrt{2}.$$

Thus we get four critical points:

$$P_1 = (3/\sqrt{2}, \sqrt{2}), P_2 = (3/\sqrt{2}, -\sqrt{2}), P_3 = (-3/\sqrt{2}, \sqrt{2}), P_4 = (-3/\sqrt{2}, -\sqrt{2}).$$

The values of  $f$  at these points are:

$$f(P_1) = 3, f(P_2) = -3, f(P_3) = -3, f(P_4) = 3.$$

The largest of these values is 3, hence 3 is the global maximum.

**Problem 10.** (20 points) Find the least distance from the origin to the surface  $g(x, y, z) = x^2y - z^2 + 9 = 0$ .

**Solution.** Minimizing the distance function is the same as minimizing the function  $f(x, y, z) = x^2 + y^2 + z^2$ . The Lagrange equation for the critical points of the constrained problem is

$$\nabla f = \lambda \nabla g, \quad x^2y - z^2 + 9 = 0.$$

Hence we get:

$$(2x, 2y, 2z) = \lambda(2xy, x^2, -2z), \quad x^2y - z^2 + 9 = 0.$$

If  $x = 0, y = 0$ , then  $z^2 = 9$  and hence  $z = \pm 3$ . Thus the first pair of critical points is  $P_1 = (0, 0, 3), P_2 = (0, 0, -3)$ . If  $z = 0$  but  $x \neq 0, y \neq 0$  then

$$x/(xy) = 2y/x^2, \quad x^2y + 9 = 0.$$

$$x^2 = 2y^2, \quad 2y^3 = -9,$$

thus  $y = -(9/2)^{1/3}$ ,  $x = \pm\sqrt{2}y = \pm\sqrt{2}(9/2)^{1/3}$ . This gives us two more critical points:

$$P_3 = (\sqrt{2}(9/2)^{1/3}, -(9/2)^{1/3}, 0), P_4 = (-\sqrt{2}(9/2)^{1/3}, -(9/2)^{1/3}, 0).$$

Finally, if  $x \neq 0, y \neq 0, z \neq 0$  we get:

$$x/(xy) = 2y/x^2 = (2z)/(-2z) = -1.$$

Hence  $y = -1$ ,  $x^2 = 2$ ,  $x = \pm\sqrt{2}$ . Substituting this in the equation  $x^2y - z^2 + 9 = 0$  we get:  $-2 - z^2 + 9 = 0$ ,  $z^2 = 7$ ,  $z = \pm\sqrt{7}$ . Thus we get four more critical points:

$$P_5 = (\sqrt{2}, -1, \sqrt{7}), P_6 = (-\sqrt{2}, -1, \sqrt{7}),$$

$$P_7 = (\sqrt{2}, -1, -\sqrt{7}), P_8 = (-\sqrt{2}, -1, -\sqrt{7}).$$

We now compare the values of  $f$  at these points:

$$f(P_1) = f(P_2) = 9,$$

$$f(P_3) = f(P_4) = 3(81/4)^{1/3} = 9\left(\frac{3}{4}\right)^{1/3} < 9 < 10.$$

$$f(P_5) = f(P_6) = f(P_7) = f(P_8) = 2 + 1 + 7 = 10.$$

Hence the minimal value of  $x^2 + y^2 + z^2$  is  $9\left(\frac{3}{4}\right)^{1/3}$ , the least distance from the origin to the surface is  $3\left(\frac{3}{4}\right)^{1/6}$ .