MATHEMATICS 2210-3. Homework 7: Solutions.

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1. Problems # 4, 8, 15 from Section 15.3.

Problem 4. (10 points) Compute the limit:

$$\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x+y+1)^2}$$

Solution. Note that

$$\lim_{(x,y)\to(-1,2)} (xy - y^3) = (-1)2 - (2)^3 = -10$$

(since $xy - y^3$ is a polynomial function of x, y and each polynomial function is continuous). Similarly:

$$\lim_{(x,y)\to(-1,2)} (x+y+1)^2 = (-1+2+1)^2 = 4 \neq 0.$$

Hence we have a limit of a ratio of two functions where both numerator and denominator have limits and the limit of the denominator is non-zero; thus

$$\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x+y+1)^2} = \frac{-10}{4} = -2.5.$$

Problem 8. (10 points) Compute the limit:

$$\lim_{(x,y)\to(-1,2)} \frac{x^4 - y^4}{x^2 + y^2}.$$

Solution. Recall that $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$, hence

$$\lim_{(x,y)\to(-1,2)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(y,x)\to(-1,2)} (x^2 - y^2) = (-1)^2 - 2^2 = -3,$$

(since $x^2 - y^2$ is a polynomial function and is therefore continuous).

Problem 15. (10 points) Show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

Solution. We first consider the limit along the x-axis:

$$\lim_{x\to 0} \frac{x0}{x^2+0^2} = 0.$$

Now consider the limit along the line $x = y$:

$$\lim_{x\to 0} \frac{xx}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = 1/2 \neq 0.$$

The limits are different, therefore the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

2. # 6, 12 from Section 15.4.

Problem 6. (10 points) Compute the gradient of the function:

$$f(x,y) = \sin^3(x^2y).$$

Solution. By the chain rule we have:

$$\frac{\partial f}{\partial x} = 3\sin^2(x^2y)\cdot\cos(x^2y)\cdot\frac{\partial(x^2y)}{\partial x} = 3\sin^2(x^2y)\cdot\cos(x^2y)(2xy).$$

Similarly,

$$\frac{\partial f}{\partial y} = 3\sin^2(x^2y)\cdot\cos(x^2y)\cdot\frac{\partial(x^2y)}{\partial y} = 3\sin^2(x^2y)\cdot\cos(x^2y)(x^2).$$

Hence

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = 3\sin^2(x^2y)\cos(x^2y)(2xy, x^2).$$

Problem 12. (10 points) Find the gradient and the tangent plane to the graph of the function $f(x,y)$ at the point $p = (2,-2)$. Here $f(x,y) = x^3y + 3xy^2$.

Solution. $f(p) = 8(-2) + 6(4) = 8$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y + 3xy^2) = 3x^2y + 3y^2.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y + 3xy^2) = x^3 + 6xy.$$

At the point $p = (2,-2)$ we get:

$$\nabla f(2,-2) = (12(-2) + 12, 8 - 24) = (-12, -16).$$

Therefore the tangent plane at p is

$$z = -12x - 16y - (-12(2) - 16(-2)) + 8 = -12x - 16y.$$