MATHEMATICS 2210-3. Homework 4: Solutions.

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1. Problems # 6, 8 from Section 14.3. In the problem # 6 find both unit normal vectors.

Problem # 6. Find the unit vectors perpendicular to the plane determined by the three points: $A = (-1, 3, 0)$, $B = (5, 1, 2)$ and $C = (4, -3, -1)$

Solution.
$\overrightarrow{AB} = (5, 1, 2) - (-1, 3, 0) = (6, -2, 2)$, $\overrightarrow{AC} = (4, -3, -1) - (-1, 3, 0) = (5, -6, -1)$. Then one normal vector to the plane is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 2 \\ 5 & -6 & -1 \end{vmatrix} = (14, 16, -26)$$

The magnitude of this vector is $2\sqrt{282}$. Hence the normal unit vectors are:

$$\overrightarrow{n} = (14, 16, -26)/(2\sqrt{282}) = \frac{1}{\sqrt{282}}(7, 8, -13)$$

and

$$-\overrightarrow{n} = \frac{1}{\sqrt{282}}(-7, -8, 13).$$

Problem # 8. Find the area of the parallelogram with $\overrightarrow{a} = (2, 2, -1)$ and $\overrightarrow{b} = (-1, 1, -4)$ as the adjacent sides.

Solution. The area of the parallelogram is the magnitude of the cross-product $\overrightarrow{a} \times \overrightarrow{b}$:

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix} = (-7, 9, 4).$$

Thus area equals $|(-7, 9, 4)| = \sqrt{146} \approx 12.08304597$.

2. Problems # 8, 18 from Section 14.4.

Problem # 8. Find the parametric and symmetric equations of the line through the given point paralel to the given vector: $P_0 = (-2, 2, -2)$ and $\overrightarrow{v} = (7, -6, 3)$.

Solution. The symmetric equation is:

$$\frac{x+2}{7} = \frac{y-2}{-6} = \frac{z+2}{3}$$

The parametric (in the vector form) equation is:

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\overrightarrow{v},$$

$$(x, y, z) = (-2, 2, -2) + t(7, -6, 3)$$

Hence $x = -2+7t$, $y = -6t+2$, $z = -2+3t$ is the parametric equation.

Problem # 18. Show that the lines

$$\frac{x-1}{-4} = \frac{y-2}{3} = \frac{z-4}{-2}, \quad \frac{x-2}{-1} = \frac{y-1}{1} = \frac{z+2}{6}$$

intersect and find the equation of the plane that they determine.

Solution. We first find the point of intersection by solving the system of four equations and 3 unknowns:

$$\begin{gathered} 3(x-1) = -4(y-2) \\ (-2)(y-2) = 3(z-4) \\ x-2 = 1-y \\ 6(y-1) = z+2 \end{gathered}$$

The solution is $x = 1, y = 2, z = 4$. Hence the lines indeed intersect (in a single point) and the point $P_0 = (1, 2, 4)$ is on the intersection of these lines. We also have two vectors along these lines: $\overrightarrow{v} = (-4, 3, -2)$ and $\overrightarrow{u} = (-1, 1, 6)$. Hence the normal vector to the plane is

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 6 \\ -4 & 3 & -2 \end{vmatrix} = (-20, -26, 1) = \overrightarrow{n}.$$

The equation of the plane is

$$-20x - 26y + z = (-20, -26, 1) \cdot (1, 2, 4) = -20 - 52 + 4 = -68.$$

3. Problems # 8, 26, 36 from Section 14.5.

Problem # 8. Compute velocity, speed, acceleration at the time $t_1 = \pi/2$ for the curve $\overrightarrow{r}(t) = (\sin(2t), \cos(3t), \cos(4t))$.

Solution. The velocity is

$$\overrightarrow{v}(t) = \overrightarrow{r}'(t) = (2\cos(2t), -3\sin(3t), -4\sin(4t)).$$

The acceleration is

$$\overrightarrow{a}(t) = \overrightarrow{r}''(t) = (-4\sin(2t), -9\sin(3t), -16\cos(4t)).$$

At time $t = \pi/2$ we get:

$$\vec{v}(\pi/2) = (-2, 3, 0), \text{ is the velocity,}$$
$$s(\pi/2) = |(-2, 3, 0)| = \sqrt{4+9} = \sqrt{13}, \text{ is the speed,}$$
$$\vec{a}(\pi/2) = (0, 0, -16), \text{ is the acceleration.}$$

Problem # 26. Find the curvature, the unit tangent vector, the principal normal and the binormal at $t = 1$ for the curve:

$$\vec{r}(t) = (t^3/3, 0, t^2/2).$$

Solution. I first give the solution done by direct calculations and afterwards, the more geometric solution.

The hard way. The velocity $\vec{r}'(t) = (t^2, 0, t)$. The acceleration is $\vec{r}''(t) = (2t, 0, 1)$. The velocity at $t = 1$ is $\vec{v} = (1, 0, 1)$ which has the magnitude $\sqrt{2}$. Hence the unit tangent vector $\vec{T}$ is

$$\vec{v}/|\vec{v}| = \frac{1}{\sqrt{t^4+t^2}}(t^2, 0, t) = \frac{1}{\sqrt{t^2+1}}(t, 0, 1)$$

and at $t = 1$ we get:

$$\vec{T}(1) = \frac{1}{\sqrt{2}}(1, 0, 1).$$

The principal normal vector is

$$\vec{N} = \vec{T}'/|\vec{T}'|.$$

Hence

$$\vec{T}'(t) = ((\frac{t}{\sqrt{t^2+1}})', 0, (\frac{1}{\sqrt{t^2+1}})') =$$
$$= (\frac{1}{\sqrt{t^2+1}} - \frac{t^2}{(t^2+1)^{3/2}}, 0, -\frac{t}{(t^2+1)^{3/2}}).$$

At $t = 1$ we get:

$$\vec{T}'(t) = (\frac{1}{2\sqrt{2}}, 0, \frac{-1}{2\sqrt{2}})$$

the magnitude of this vector is $1/2$. The curvature at $t = 1$ equals

$$k = \frac{|\vec{T}'(1)|}{|\vec{v}(1)|} = \frac{1}{2\sqrt{2}}.$$

The principal normal vector at $t = 1$ equals

$$\vec{T}'/|\vec{T}'| = \vec{N} = \frac{1}{\sqrt{2}}(1, 0, -1).$$

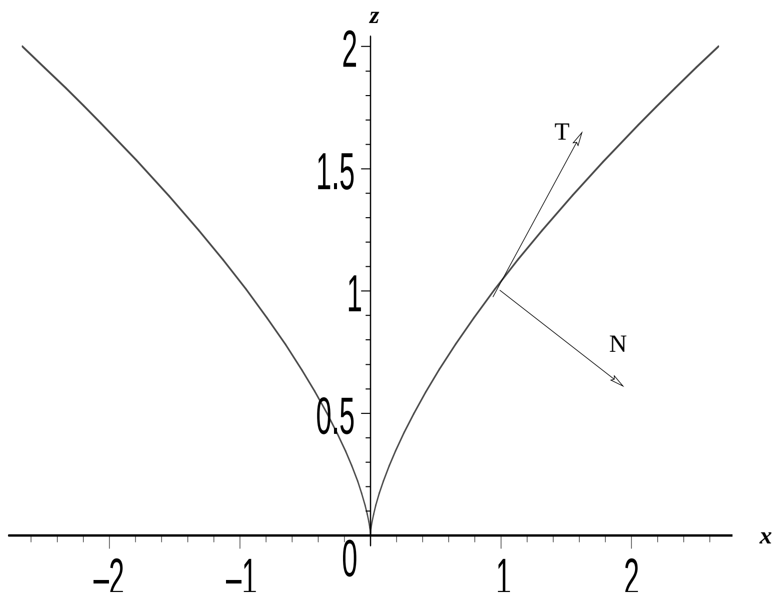
To find the binormal vector we compute the cross-product:

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{2}{\sqrt{2}} \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (0,1,0).$$

Hence $\overrightarrow{B} = (0,1,0)$.

The easy way. Note that the curve under consideration lies in the xz-plane. Hence both velocity, acceleration and the curvature vector are in the xz-pane. Thus both vectors $\overrightarrow{T}$ and $\overrightarrow{N}$ are in the xz-plane. The binormal vector $\overrightarrow{B}$ is orthogonal to both, hence it is the unit vector parallel to the y-axis, it is either $(0,1,0)$ or $(0,-1,0)$ (we just have to decide which one). Also note that the principal normal vector is the same as the normal vector $\overrightarrow{N}$ computed for the planar curve, it is the unit vector with the tail at $P(t=1) = (1/3, 0, 1/2)$ directed into the convex side of the curve.



The velocity at $t=1$ is $\overrightarrow{v} = (1,0,1)$ which has the magnitude $\sqrt{2}$. Hence the unit tangent vector $\overrightarrow{T}$ at $t=1$ is:

$$\overrightarrow{T}(1) = \frac{1}{\sqrt{2}}(1,0,1).$$

Plot it in the xz-plane. The vectors in xz-plane orthogonal to the vector $\frac{1}{\sqrt{2}}(\overrightarrow{i} + \overrightarrow{k})$ are $\frac{\pm 1}{\sqrt{2}}(\overrightarrow{i} - \overrightarrow{k})$, we just have to figure out weather to take $+$ or $-$. By plotting on the graph we see that the vector $\frac{1}{\sqrt{2}}(\overrightarrow{i} - \overrightarrow{k})$ with tail at the point $t=1$, $(x,z) = (1/3, 1/2)$ is directed into the convex side of the curve (the south-east direction). Hence we should take $+$. Now it remains to compute the cross-product to find the direction of the binormal vector:

$$\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = (0,1,0).$$

Remark. *Actually, you can figure it out without determinants. Recall that* $\overrightarrow{B} = \pm(0,1,0)$. *The vector* $\overrightarrow{N}$ *is in the counter-clockwise direction from* $\overrightarrow{T}$ *if we are looking at the xz-plane from the point* $(0,1,0)$. *Hence the cross-product is* $(0,1,0)$.

Thus (at t = 1) we have the unit tangent vector $\overrightarrow{T}(1) = \frac{1}{\sqrt{2}}(1,0,1)$, *the principal normal vector* $\overrightarrow{N}(1) = \frac{1}{\sqrt{2}}(1,0,-1)$ *and the binormal vector* $\overrightarrow{B} = (0,1,0)$.

It remains to compute the curvature at $t = 1$ (here $\overrightarrow{v}$ is the velocity vector):

$$k = \frac{|x''z' - x'z''|}{|\overrightarrow{v}|^3} = \frac{|2-1|}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

Hence $k = \frac{1}{2\sqrt{2}}$.

Problem 36. Find tangential and normal components of the acceleration vector for $\overrightarrow{r}(t) = (t, t^2, t^3)$.

Solution. $\overrightarrow{r}'(t) = (1, 2t, 3t^2)$, $\overrightarrow{r}''(t) = (0, 2, 6t)$. Their cross product is

$$\overrightarrow{r}' \times \overrightarrow{r}'' = (6t^2, -6t, 2).$$

The magnitude of this vector is

$$|\overrightarrow{r}' \times \overrightarrow{r}''| = \sqrt{36t^4 + 36t^2 + 4}.$$

Then the tangential component of the acceleration is

$$a_T = \frac{\overrightarrow{r}' \cdot \overrightarrow{r}''}{|r'|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$

The normal component is

$$a_N = \frac{|\overrightarrow{r}' \times \overrightarrow{r}''|}{|r'|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}}.$$