MATHEMATICS 2210-3. Homework 3: Solutions.

January 22, 2001

Each problem is worth 10 points.

1. Problems # 4, 31 from Section 13.5.

Problem # 4. Find the unit tangent vector $\overrightarrow{T}$ and the curvature k at the point where $t = 1$. The curve is given by the equation:

$$\overrightarrow{r}(t) = (t^3/3, t^2/2).$$

Solution. $\overrightarrow{r}'(t) = (t^2, t)$, this vector has the magnitude $\sqrt{2}$ at $t = 1$. Hence $\overrightarrow{T}(1) = \frac{1}{\sqrt{2}}(1, 1)$. Let's compute the second derivative: $\overrightarrow{r}''(t) = (2t, 1)$. Thus

$$k(1) = \frac{|x'(1)y''(1) - x''(1)y'(1)|}{|\overrightarrow{r}'(1)|^3} = \frac{|1-2|}{\sqrt{2}^3} = \frac{1}{2\sqrt{2}}.$$

Problem # 31. Find the point on the curve where the curvature is at maximum:

$$y = \cosh(x).$$

Solution.

$$k(x) = \frac{|y''|}{(1 + (y')^2)^{3/2}}.$$

Recall that $\cosh''(x) = \cosh(x) > 0$, $\cosh'(x) = \sinh(x)$, $1 + \sinh^2(x) = \cosh^2(x)$. Hence

$$k(x) = \frac{\cosh(x)}{(1 + \sinh^2(x))^{3/2}} = \frac{\cosh(x)}{\cosh^3(x)} = \frac{1}{\cosh^2(x)}.$$

Since $\cosh(x) > 0$ for each x, the function $\frac{1}{\cosh^2(x)}$ attains its maximum at the point where the function $\cosh(x)$ attains its minimum. To find the minimum of $\cosh(x)$ consider

$$0 = \cosh'(x) = \sinh(x).$$

The only point where $\sinh(x)$ attains its zero is $x = 0$. We also have $\cosh'(x) = \sinh(x) < 0$ for $x < 0$ and $\cosh'(x) = \sinh(x) > 0$ for $x > 0$, hence $x = 0$ is the point where $\cosh(x)$ is at its minimum. Thus the answer is $x = 0$.

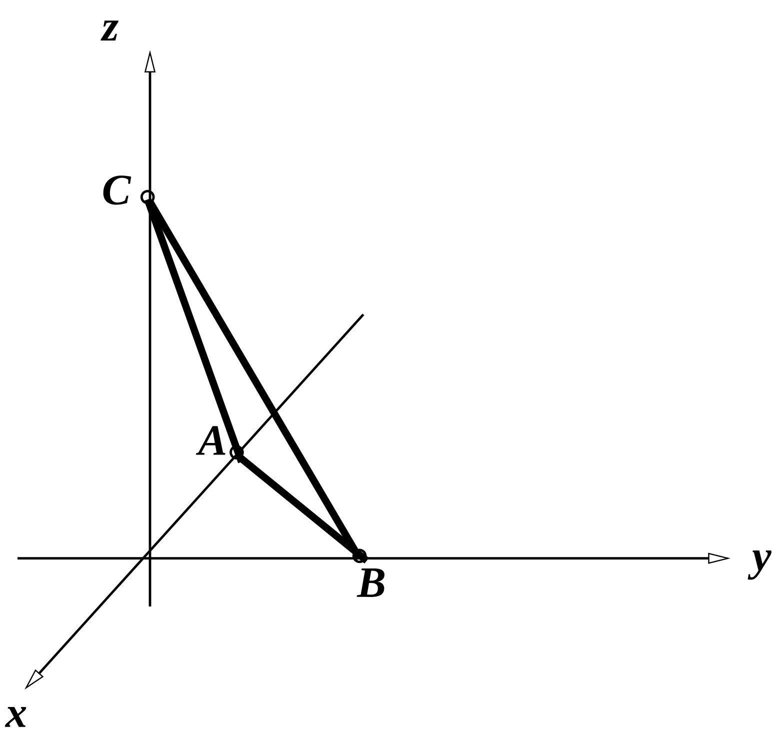
2. Problems # 6, 20 from Section 14.1.

Problem # 6. Show that $A = (4, 5, 3)$, $B = (1, 7, 4)$ and $C = (2, 4, 6)$ are vertices of an equilateral triangle.

Solution. $|AB| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$. $|AC| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$. $|BC| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Hence all the sides of the triangle have the same lenght and thus the triangle is equilateral.

Problems # 20. Sketch the graph of the given equation. $-3x + 2y + z = 6$.

Solution. This is the equation of a plane in the 3-space. First find the intersection points of the graph with the coordinate lines: $y = 0 = z$ implies that $x = -2$. $x = 0 = z$ implies that $y = 3$. $x = y = 0$ implies that $z = 6$. Thus the intersection points are $A = (-2, 0, 0)$, $B = (0, 3, 0)$ and $C = (0, 0, 6)$.



3. Problems # 28, 32 from Section 14.2.

Problem # 28. Find equation of the plane through $P_0 = (-1, 2, -3)$ parallel to the plane $2x + 4y - z = 6$.

Solution. Parallel planes have same normal vectors, hence as the normal vector for our plane we can take $\overrightarrow{n} = (2, 4, -1)$ (the normal vector of the given plane). Equation of the plane is $\overrightarrow{OP} \cdot \overrightarrow{n} = \overrightarrow{OP_0} \cdot \overrightarrow{n}$:

$$2x + 4y - z = (-1, 2, -3) \cdot (2, 4, -1) = -2 + 8 + 3 = 9.$$

Thus the equation of the plane is $2x + 4y - z = 9$.

Problem # 32. Find the distance bwteen the paralel planes $-3x + 2y + z - 9 = 0$ and $6x - 4y - 2z - 19 = 0$.

Solution. First find a point on the first plane, by taking $x = y = 0$, then $z = 9$. Hence we get the point $P_0 = (0, 0, 9)$. Then the distance from P_0 to the second plane (which is the same as the distance between the planes) is

$$\frac{|6 \cdot 0 - 4 \cdot 0 - 2 \cdot 9 - 19|}{\sqrt{6^2 + 4^2 + 2^2}} = \frac{37}{\sqrt{36 + 16 + 4}} = \frac{37}{\sqrt{56}}.$$