

MATHEMATICS 2210-3. Homework 12: Solution.

April 2, 2001

1. Problems # 2, 6 from Section 16.5. Each problem is worth 20 points.

Problem 2. Compute the mass M and the center of mass of the lamina S bounded by the curves $y = 0$, $y = \sqrt{4 - x^2}$ where the density of the lamina is $\delta(x, y) = y$.

Solution. The graph $y = \sqrt{4 - x^2}$ is the upper semi-circle of the circle of radius 2 centered at the origin. Thus the lamina S is the upper half-disk of the radius 2 centered at the origin. We compute the integrals using the polar coordinates.

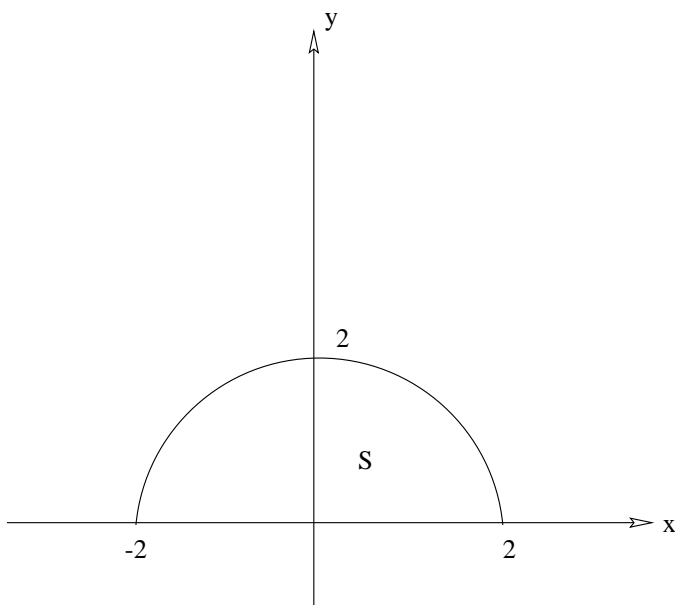


Figure 1:

$$M = \iint_S y dx dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} r^2 \sin(\theta) dr d\theta =$$
$$\int_{r=0}^{r=2} r^2 dr \int_{\theta=0}^{\theta=\pi} \sin(\theta) d\theta = [r^3/3]_0^2 [-\cos(\theta)]_0^\pi = 8/3 \cdot 2 = 16/3.$$

Similarly,

$$M_x = \iint_S xy dx dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} r^3 \sin(\theta) \cos(\theta) dr d\theta =$$

$$\int_0^2 r^3 dr \int_0^\pi \sin(\theta) \cos(\theta) d\theta = [r^4/4]_0^2 \left[\frac{1}{2} \sin^2(\theta) \right]_0^\pi = 4 \cdot \frac{1}{2} \cdot 0 = 0.$$

(You can also notice that the equality $M_x = 0$ follows from the symmetry in y -axis.)

$$M_y = \iint_S y^2 dx dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} r^3 \cos^2(\theta) dr d\theta =$$

$$\int_0^2 r^3 dr \int_0^\pi \cos^2(\theta) d\theta = [r^4/4]_0^2 \cdot \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right]_0^\pi = 4\pi/2 = 2\pi.$$

The center of mass equals

$$(\bar{x}, \bar{y}) = (M_x/M, M_y/M) = (0, 2\pi/(16/3)) = (0, 3\pi/8).$$

Problem 6. Compute the mass M and the center of mass of the lamina S bounded by the curves $y = 0$, $y = e^x$, $x = 0$, $x = 1$ where the density of the lamina is $\delta(x, y) = 2 - x + y$.

Solution. See Figure 2 for the sketch of the lamina.

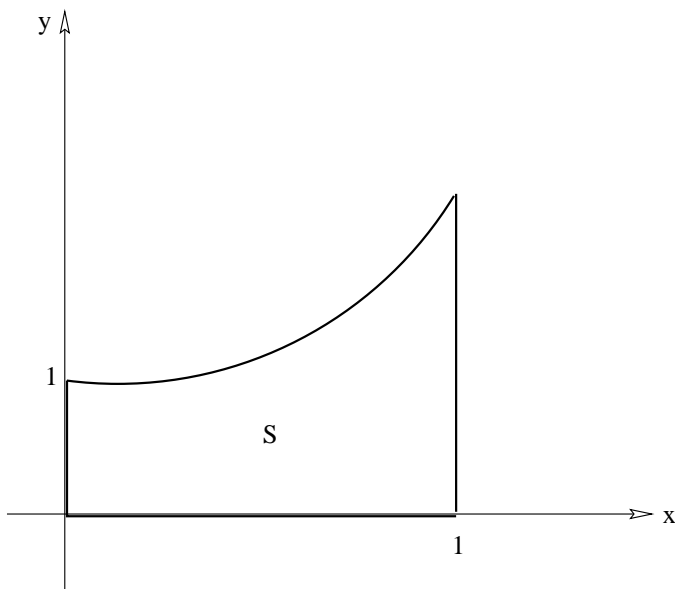


Figure 2:

$$\begin{aligned} M &= \iint_S \delta(x, y) dx dy = \int_{x=0}^{x=1} \int_{y=0}^{y=e^x} [2 - x + y] dy dx = \int_{x=0}^{x=1} [(2 - x)y + y^2/2]_{y=0}^{y=e^x} dx = \\ &= \int_0^1 ((2 - x)e^x + e^{2x}/2) dx = \int_0^1 (2e^x + e^{2x}/2) dx - \int_0^1 xe^x dx \end{aligned}$$

The first integral equals

$$[2e^x + e^{2x}/4]_0^1 = 2e + e^2/4 - 2 - 1/4 = 2e + e^2/4 - 2.25.$$

The second integral can be computed using the integration by parts:

$$\int_0^1 xe^x dx = \int_0^1 xde^x = [xe^x]_0^1 - \int_0^1 e^x dx = e - [e^x]_0^1 = e - e + 1 = 1.$$

Hence

$$M = \int_0^1 (2e^x + e^{2x}/2)dx - \int_0^1 xe^x dx = 2e + e^2/4 - 2.25 - 1 = 2e + e^2/4 - 3.25.$$

We now compute the center of mass.

$$M_x = \iint_S x\delta(x, y)dxdy = \int_{x=0}^{x=1} \int_{y=0}^{y=e^x} x(2-x+y)dydx =$$

$$\int_{x=0}^{x=1} [(2x-x^2)y + xy^2/2]_{y=0}^{y=e^x} dx = \int_0^1 ((2x-x^2)e^x + xe^{2x}/2)dx$$

We now compute the three integrals:

$$\int_0^1 2xe^x dx = [2xe^x]_0^1 - 2 \int_0^1 e^x dx = 2e - 2(e-1) = 2.$$

$$\int_0^1 x^2e^x dx = [x^2e^x]_0^1 - \int_0^1 2xe^x dx = e - 2.$$

$$\int_0^1 xe^{2x}/2 dx = [xe^{2x}/4]_0^1 - \int_0^1 e^{2x}/4 dx = e^2/4 - [e^{2x}/8]_0^1 = e^2/4 - e^2/8 + 1/8 = \frac{1}{8}e^2 + \frac{1}{8}.$$

Thus

$$M_x = 2 - (e-2) + \frac{1}{8}e^2 + \frac{1}{8} = \frac{33}{8} - e + \frac{1}{8}e^2.$$

$$M_y = \iint_S y\delta(x, y)dxdy = \int_{x=0}^{x=1} \int_{y=0}^{y=e^x} y(2-x+y)dydx =$$

$$\int_{x=0}^{x=1} [(2-x)y^2/2]_0^{e^x} + [y^3/3]_0^{e^x} dx =$$

$$\int_{x=0}^{x=1} (2-x)e^{2x}/2 + e^{3x}/3 dx = \int_0^1 e^{2x} dx - \frac{1}{2} \int_0^1 xe^{2x} dx + \frac{1}{3} \int_0^1 e^{3x} dx.$$

We now compute the three integrals:

$$\int_0^1 e^{2x} dx = \frac{1}{2}[e^{2x}]_0^1 = \frac{1}{2}(e^2 - 1).$$

$$\frac{1}{2} \int_0^1 xe^{2x} dx = \frac{1}{8}e^2 + \frac{1}{8}.$$

$$\frac{1}{3} \int_0^1 e^{3x} dx = \frac{1}{9}(e^3 - 1).$$

Thus

$$M_y = \frac{1}{2}(e^2 - 1) - \frac{1}{8}e^2 - \frac{1}{8} + \frac{1}{9}(e^3 - 1) = \frac{3}{8}e^2 + \frac{1}{9}e^3 - \frac{53}{72}.$$

The center of mass equals

$$(\bar{x}, \bar{y}) = (M_x/M, M_y/M) = \left(\frac{\frac{33}{8} - e + \frac{1}{8}e^2}{2e + 0.25e^2 - 3.25}, \frac{\frac{3}{8}e^2 + \frac{1}{9}e^3 - \frac{53}{72}}{2e + 0.25e^2 - 3.25} \right).$$

2. Problems # 2, 4 from Section 16.6.

Problem 2. (10 points) Compute area of the part of the plane $3x - 2y + 6z = 12$ that is bounded by the planes $x = 0$, $y = 0$ and $3x + 2y = 12$.

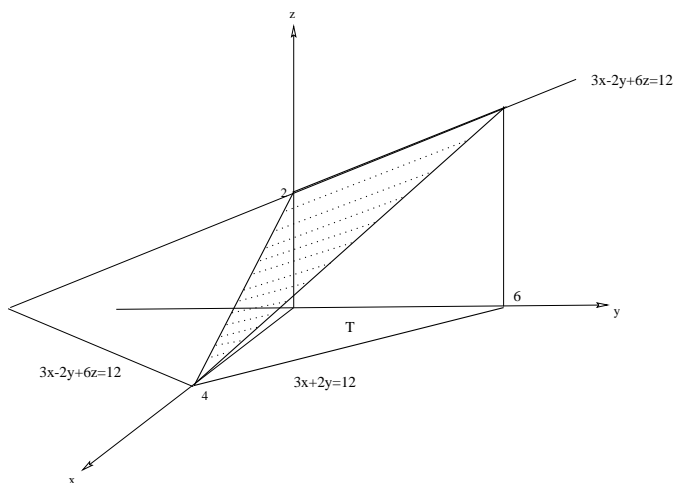


Figure 3:

Solution. The surface whose area we have to compute projects to the triangle T in xy -plane with the vertices $(0, 0)$, $(4, 0)$, $(6, 0)$. See Figure 3. The plane $3x - 2y + 6z = 12$ is the graph of the function

$$f(x, y) = 2 - x/2 + y/3.$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1/4 + 1/9 + 1} = \frac{7}{6}$$

Note that $Area(T) = 4 \cdot 6/2 = 12$. Hence

$$Area = \frac{7}{6} Area(T) = \frac{12 \cdot 7}{6} = 14.$$

Problem 4. (20 points) Compute area of the part of the surface $z = \sqrt{4 - y^2}$ in the first octant that is directly above the disk $x^2 + y^2 \leq 4$ in the xy -plane.

Remark. Actually, the textbook is asking for area of the part of the surface $z = \sqrt{4 - y^2}$ in the first octant that is directly above the **circle** $x^2 + y^2 = 4$ in the xy -plane. This part of the surface is actually a curve, hence its area is zero. Everybody who gave this answer should get the full credit for this problem, I think however that this was a misprint.

Solution. Let $R = \{x, y | x > 0, y > 0, x^2 + y^2 \leq 4\}$. $f(x, y) = \sqrt{4 - y^2}$, $f_x = 0$,

$$f_y = \frac{-y}{\sqrt{4 - y^2}}$$

$$\begin{aligned}
 \text{Area} &= \iint_R \sqrt{1 + f_y^2} dx dy = \iint_R \sqrt{1 + \frac{y^2}{4 - y^2}} dx dy = \\
 &= \iint_R \sqrt{\frac{4}{4 - y^2}} dx dy = \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} \sqrt{\frac{4}{4 - y^2}} dx dy = \int_{y=0}^{y=2} 2 dy = 4.
 \end{aligned}$$

3. Problems # 2, 4, 10 from Section 16.7.

Problem 2. (10 points) Evaluate the iterated integral

$$\int_{-3}^7 \int_0^{2x} \int_y^{x-1} dz dy dx.$$

Solution.

$$\int_y^{x-1} dz = (x - 1) - y.$$

$$\int_0^{2x} ((x - 1) - y) dy = [-y^2/2 + (x - 1)y]_0^{2x} = -2x^2 + 2x(x - 1) = -2x.$$

$$\int_{-3}^7 \int_0^{2x} \int_y^{x-1} dz dy dx = \int_{-3}^7 -2x dx = [-x^2]_{-3}^7 = -49 + 9 = -40.$$

Note that this integral is negative: it DOES NOT compute the volume of a solid (because of the bounds of integration)!

Problem 4. (10 points) Evaluate the iterated integral

$$\int_0^5 \int_{-2}^4 \int_1^2 6xy^2z^3 dx dy dz.$$

Solution.

$$\begin{aligned}
 \int_0^5 \int_{-2}^4 \int_1^2 6xy^2z^3 dx dy dz &= \int_0^5 z^3 dz \int_{-2}^4 3y^2 dy \int_1^2 2x dx = \\
 &= [z^4/4]_0^5 [y^3]_{-2}^4 [x^2]_1^2 = (625/4)(64 + 8)(4 - 1) = 33750.
 \end{aligned}$$

Problem 10. (10 points) Sketch the solid. Write the iterated integral for

$$\iiint_S f(x, y, z) dV,$$

where $S = \{(x, y, z) : 0 \leq x \leq \sqrt{4 - y^2}, 0 \leq y \leq 2, 0 \leq z \leq 3\}$.

Solution. To sketch the solid note that this is the region between the planes $z = 0, z = 3$ and directly above the quarter of the disk $x^2 + y^2 \leq 4$ which is in the first quadrant (in xy -plane). See Figure 4.

$$\iiint_S f(x, y, z) dV = \int_0^3 \int_0^2 \int_0^{\sqrt{4-y^2}} f(x, y, z) dx dy dz.$$

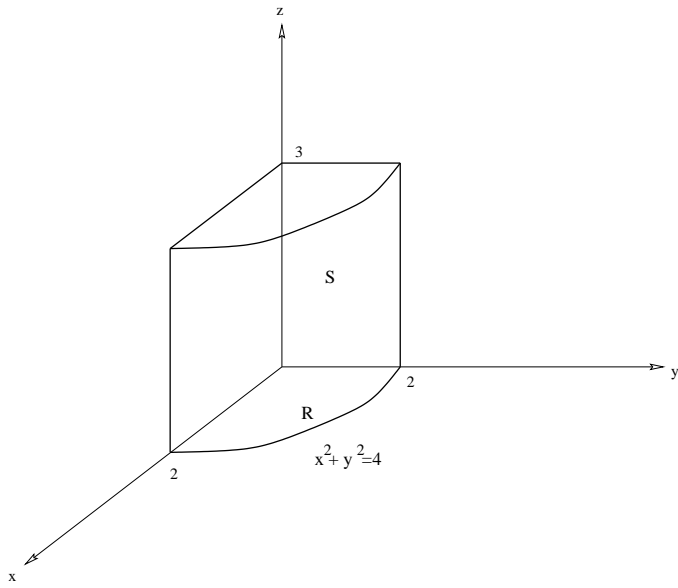


Figure 4: