MATHEMATICS 2210-3. Homework 1: Solution.

January 18, 2001

Each problem is worth 10 points.

1. Problems # 7, 17, 29, 43, 50 from Section 13.1.

In # 7 and # 17 graph the curve with the given parametric representation. Is this curve closed? Is it simple? Obtain the Cartesian equation of the curve by eliminating the parameter.

Problem 7. $x = 1/s$, $y = s$, $1 \leq s < 10$.

Solution. $y = 1/x$, $0.1 < x \leq 1$ is a simple nonclosed curve, which is a part of the hyperbola.

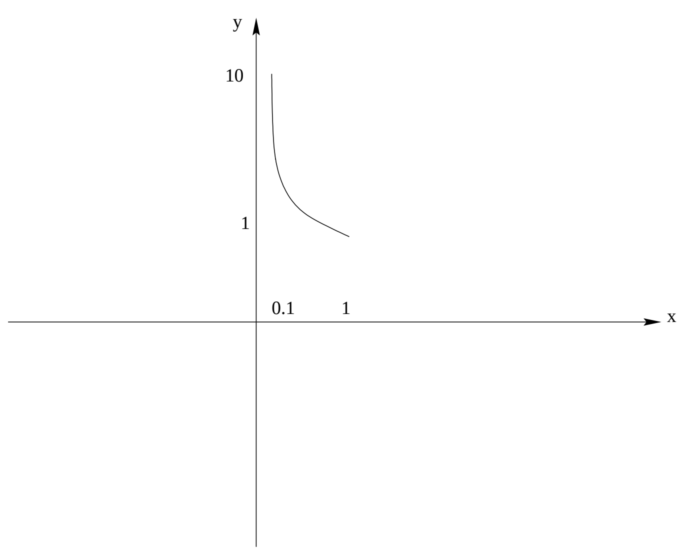

Figure 1: *Problem 7.*

Problem 17. $x = 9\sin^2(\theta)$, $y = 9\cos^2(\theta)$, $0 \leq \theta \leq \pi$.

Solution. $x + y = 9\sin^2(\theta) + 9\cos^2(\theta) = 9$. Hence the curve lies on the straight line $x + y = 9$. For $0 \leq \theta \leq \pi$ the coordinates x and y vary between 0 and 9. If $\theta = 0$ then we get one extreme point of the curve: $(x, y) = (0, 9)$. If $\theta = \pi/2$ we get the other extreme point of the curve: $(x, y) = (9, 0)$. The curve is closed since $(x(0), y(0)) = (x(\pi), y(\pi)) = (0, 9)$, however the curve is not simple since

$$(x(\pi - \theta), y(\pi - \theta)) = (x(\theta), y(\theta))$$

for any θ.

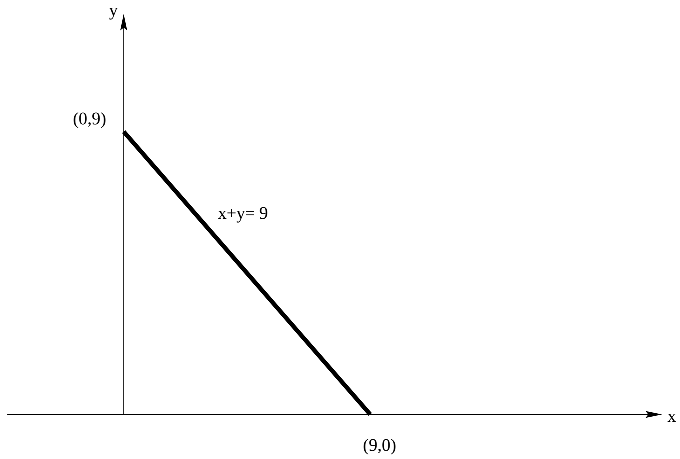

Figure 2: *Problem 17.*

Problem 29. Find dy/dx and d^2y/dx^2 without eliminating the parameter:

$$x = \frac{1}{1+t^2}, y = \frac{1}{t(1-t)}, 0 < t < 1.$$

Solution. Note that $\frac{dx}{dt} = \frac{-2t}{(1+t^2)^2}$. Hence

$$g(t) = \frac{dy}{dx} = \frac{dy}{dt}/(\frac{dx}{dt}) =$$

$$\frac{-1+2t}{t^2(1-t)^2} : \frac{-2t}{(1+t^2)^2} = \frac{(1-2t)(1+t^2)^2}{2t^3(1-t)^2}.$$

Now let's compute the second derivative:

$$\frac{d^2y}{dx^2} = \frac{dg}{dx} = \frac{dg}{dt}/(\frac{dx}{dt}).$$

Since we already know that $\frac{dx}{dt} = \frac{-2t}{(1+t^2)^2}$, it remains to compute

$$\frac{dg}{dt} = \frac{(1+t^2)(3t^3+7t^2-9t+3)}{2t^4(-1+t)^3} = \frac{3t^5+7t^4-6t^3+10t^2-9t+3}{2t^4(-1+t)^3}.$$

Hence

$$\frac{d^2y}{dx^2} = \frac{dg}{dt}/(\frac{dx}{dt}) = \frac{(1+t^2)^3(3t^3+7t^2-9t+3)}{-4t^5(-1+t)^3} = \frac{(3t^5+7t^4-6t^3+10t^2-9t+3)(1+t^2)^2}{4t^5(1-t)^3}.$$

Problem 43. Find the length of the parametric curve defined on the given interval:

$$x = 4\sqrt{t}, y = t^2 + \frac{1}{2t}, \frac{1}{4} \le t \le 1.$$

Solution. $x'(t) = 2/\sqrt{t}$, $y'(t) = 2t - \frac{1}{2t^2}$. Length equals

$$\int_{0.25}^{1} \sqrt{(x')^2 + (y')^2} dt = \int_{0.25}^{1} \sqrt{4/t + (2t - \frac{1}{2t^2})^2} dt = \int_{0.25}^{1} \sqrt{4/t + 4t^2 - 2/t + \frac{1}{4t^4}} =$$

$$= \int_{0.25}^{1} \sqrt{(2t + \frac{1}{2t^2})^2} dt = \int_{0.25}^{1} (2t + \frac{1}{2t^2}) dt = \left[t^2 - \frac{1}{2t} \right]_{0.25}^{1} = 1 - 1/2 - (1/16 - 2) =$$

$$= 39/16 = 2.44.$$

Problem 50. Find area of the surface generated by revolving the curve $x = \cos(t), y = 3 + \sin(t)$ for $0 \leq t \leq 2\pi$ around the x-axis.

Solution. First note that this curve is the unit circle with the center at $(0, 3)$. The first solution of the problem is an application of the Pappus theorem: the area equals the length of the curve that is revolved (which is (2π)) multiplied by the distance traveled by the center of the curve (which is 6π), hence area equals $12\pi^2$.

Another solution:

$$Area = \int_0^{2\pi} 2\pi y(t) \sqrt{(x')^2 + (y')^2} dt =$$

$$\int_0^{2\pi} 2\pi \cdot (3 + \sin(t)) \sqrt{\sin^2(t) + \cos^2(t)} dt =$$

$$\int_0^{2\pi} (6\pi + 2\pi \sin(t)) dt = \left[6\pi t - 2\pi \cos(t) \right]_0^{2\pi} = 12\pi^2.$$

2. Problems # 3, 6 from Section 13.2.

Problem 3. Draw the vector $\overrightarrow{w} = \overrightarrow{u_1} + \overrightarrow{u_2} + \overrightarrow{u_3}$.

Solution. See Figure 3.

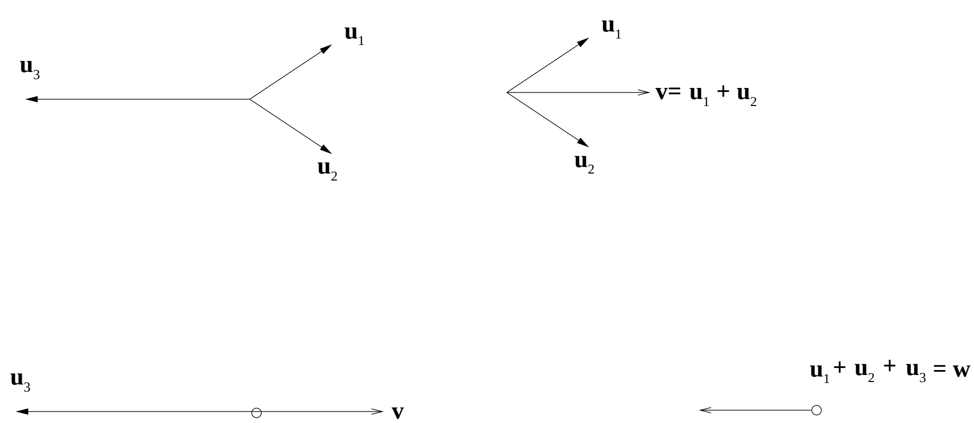

Figure 3: *Problem 3.*

Problem 6. In the large triangle $\overrightarrow{m}$ is the median (it bisects the side to which it is drawn). Express $\overrightarrow{m}$ and $\overrightarrow{n}$ in terms of $\overrightarrow{u}$ and $\overrightarrow{v}$. See Figure 4.

Solution.

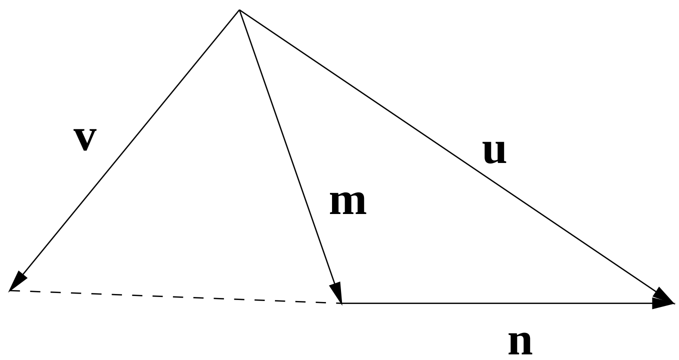

Figure 4: *Problem 6.*

$\overrightarrow{m} = (\overrightarrow{u} + \overrightarrow{v})/2$ since $\overrightarrow{m}$ is one half of the diagonal (with the tail at the same point as the tails of $\overrightarrow{u}$ and $\overrightarrow{v}$) in the paralelogram formed by $\overrightarrow{u}$ and $\overrightarrow{v}$. Similarly, $\overrightarrow{n} = (\overrightarrow{v} - \overrightarrow{u})/2$ since $\overrightarrow{n}$ is half the other diagonal in the same parallelogram, = half of the directed segment connecting the head of $\overrightarrow{v}$ with the head of $\overrightarrow{v}$.

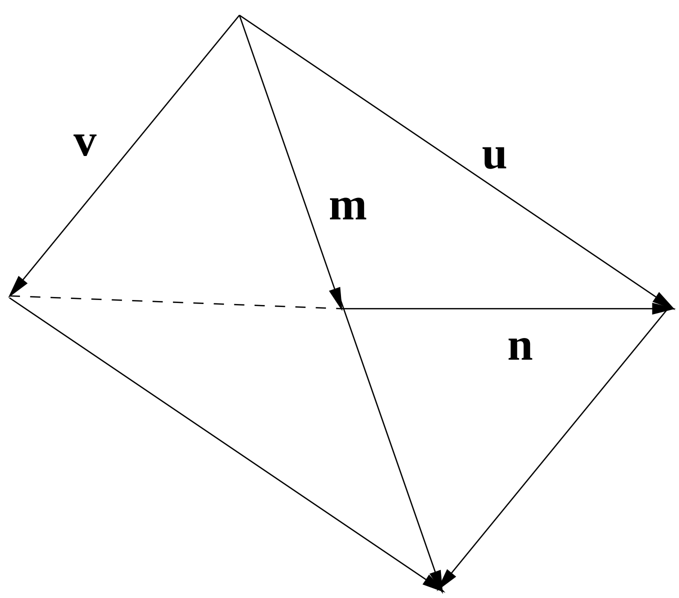

Figure 5: *Solution of Problem 6.*