MATHEMATICS 2210-3. Homework 13: Solution.

April 15, 2001

1. Problems # 4, 12 from Section 16.8. Each problem is worth 20 points.

Problem 4. Compute the volume of the solid under the surface $z = xy$ above the xy-plane within the cylinder $x^2 + y^2 = 2x$. Use the cylindrical coordinates.

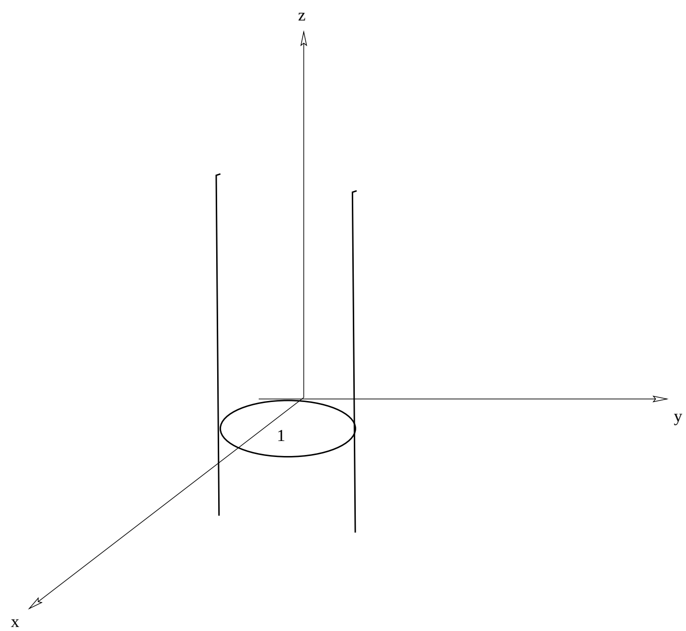

Figure 1:

Solution. xy is positive for $x > 0, y > 0$ and $x < 0, y < 0$. Hence the projection of the solid to the xy-plane is contained in the 1-st and 3-rd coordinate quadrants. The equation $x^2 + y^2 = 2x$ is equivalent to $(x-1)^2 + y^2 = 1$, which is the circle with the center $(1, 0)$ and radius 1. Thus the cylinder $x^2 + y^2 = 2x$ in the 3-space is the circular cylinder over the circle $(x-1)^2 + y^2 = 1$. The interior of this cylinder is given by the inequality $(x-1)^2 + y^2 \leq 1$. Thus the projection R of the solid to the xy-plane is the upper half of the disk $(x-1)^2 + y^2 \leq 1$ (the one, contained in the 1-st quadrant). The volume is given (in the cartesian coordinates) by the iterated integral

$$\int_{x=0}^{x=2} \int_{y=0} \sqrt{2x - x^2} \int_0^{xy} dz dy dx.$$

In cylindrical coordinates we will have the following description of the half-disk R:

$$0 \le \theta \le \pi/2, r^2 \le 2r\cos(\theta),$$

Equivalently:

$$0 \le \theta \le \pi/2, r \le 2\cos(\theta).$$

The whole solid S in the cylindrical coordinates is:

$$0 \le z \le xy = r^2\cos(\theta)\sin(\theta) = \frac{r^2}{2}\sin(2\theta),$$

$$0 \le \theta \le \pi/2, r \le 2\cos(\theta).$$

Thus the volume equals

$$\int_0^{\pi/2}\int_0^{2\cos(\theta)}\int_0^{r^2\cos(\theta)\sin(\theta)} r dz dr d\theta =$$

$$\int_0^{\pi/2}\int_0^{2\cos(\theta)} r^3\cos(\theta)\sin(\theta) dr d\theta =$$

$$\int_0^{\pi/2}\frac{(2\cos(\theta))^4}{4}\cos(\theta)\sin(\theta)d\theta =$$

$$\int_0^{\pi/2} 4\cos^5(\theta)\sin(\theta)d\theta = \frac{2}{3}[-\cos^6(\theta)]_0^{\pi/2} = 2/3.$$

Problem 12. Compute the volume of the solid within the sphere $x^2+y^2+z^2 = 16$ outside of the cone $z = x^2+y^2$ and above the xy-plane. Use the spherical coordinates.

Solution. The volume is the different of the two volumes: $V = V_1 - V_2$ where V_1 is the volume of the upper half ($z \ge 0$) of the ball $x^2+y^2+z^2 \le 16$, V_2 is the volume of the conical sector $z \ge x^2+y^2$ within the ball $x^2+y^2+z^2 \le 16$. Note that the angle between the cone $z = x^2+y^2$ and the z-axis is $\pi/4$ (this cone is the surface of revolution obtained by revolving the line $z = x$ along the z-axis).

In the spherical coordinates the first volume is

$$V_1 = \int_{\rho=0}^{\rho=4}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\rho^2\sin(\phi)d\phi d\theta d\rho =$$

$$\int_0^4\int_0^{2\pi}\rho^2 d\theta d\rho = 2\pi 4^3/3 = \frac{128\pi}{3}.$$

In the spherical coordinates the second volume equals

$$V_2 = \int_{\rho=0}^{\rho=4}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\rho^2\sin(\phi)d\phi d\theta d\rho =$$

$$\frac{2\pi}{3}[\rho^3]_0^4[-\cos(\phi)]_0^{\pi/4} = \frac{2\pi}{3}64(1-\sqrt{2}2) = \frac{64\pi(2-\sqrt{2})}{3}.$$

Thus

$$V = V_1 - V_2 = \frac{128\pi}{3} - \frac{64\pi(2-\sqrt{2})}{3} = \frac{64\sqrt{2}\pi}{3}.$$

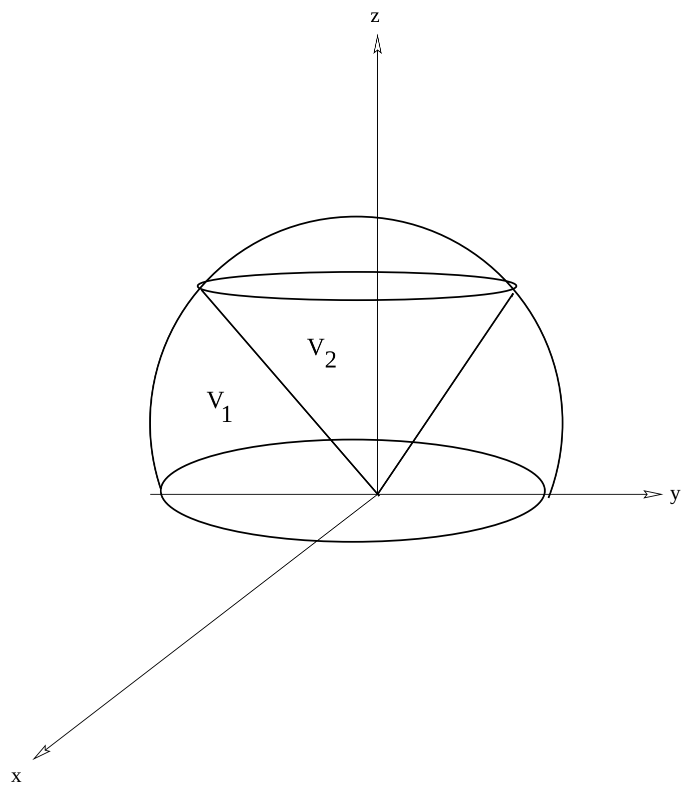

Figure 2:

2. Problems # 14, 20 from Section 17.1.

Problem 14. (5 points) Compute *curl* and *div* for the vector field $F(x,y,z) = (x^2, y^2, z^2)$.

Solution. $curl(F) = 0$. $div(F) = 2x + 2y + 2z$.

Problem 20. (20 points) Show that

(a) $div(curl \overrightarrow{F}) = 0$.

(b) $curl(grad f) = 0$.

(c) $div(f\overrightarrow{F}) = f div(\overrightarrow{F}) + (grad f) \cdot \overrightarrow{F}$.

(d) $curl(f\overrightarrow{F}) = f curl(\overrightarrow{F}) + (grad f) \times \overrightarrow{F}$. (This part we have done in the class.)

Here $\overrightarrow{F} = (F_1, F_2, F_3)$ is a vector field in $\mathbb{R}^3$ with continuous 2-nd partial derivatives.

Solution. (a) The formal reason for this identity is $div(curl \overrightarrow{F}) = \nabla \cdot (\nabla \times \overrightarrow{F}) = 0$ since $\nabla \times \overrightarrow{F}$ is "orthogonal to both ∇ and $\overrightarrow{F}$" (the latter is true for the ordinary vectors, but ∇ is not!). Here is the actual computation:

$$div(curl \overrightarrow{F}) =$$

$$\frac{\partial}{\partial x}(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) =$$

$$[\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}] + [-\frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial z \partial x}] + [\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}] = 0.$$

(b) The formal comutational reason for this identity is $curl(grad) = \nabla \times \nabla = 0$.

Here is the actual computation:

$$grad(f) = (F_1, F_2, F_3) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}).$$

$$curl(gradf) = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}) =$$

$$(f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) = (0,0,0).$$

(c) Let's compare the lefthand side with the righthand side:

$$div(f\overrightarrow{F}) = (fF_1)_x + (fF_2)_y + (fF_3)_z = (f_xF_1 + f_yF_2 + f_zF_3) + f(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}).$$

$$(gradf)\cdot \overrightarrow{F} + fdiv(\overrightarrow{F}) = (f_xF_1 + f_yF_2 + f_zF_3) + f(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}).$$

Hence LHS=RHS and we are done.

3. Problems # 6, 10 from Section 17.2.

Problem 6. (10 points) Compute the line integral

$$\int_C (x^2 + y^2 + z^2)ds$$

where C is the curve $x = 4\cos(t), y = 4\sin(t), z = 3t,\ 0 \leq t \leq 2\pi$.

Solution.

$$\int_C (x^2+y^2+z^2)ds = \int_0^{2\pi} (16\cos^2(t)+16\sin^2(t)+9t^2)\sqrt{16\sin^2(t) + 16\cos^2(t) + 9}dt =$$

$$\int_0^{2\pi} 5(16+9t^2)dt = 5[16t + 3t^3]_0^{2\pi} = 5[32\pi + 24\pi^3]$$

Problem 10. (10 points) Compute the line integral

$$\int_C y^3dx + x^3dy$$

where C is the curve $x = 2t, y = t^2 - 3, z = 3t,\ -2 \leq t \leq 1$.

Solution.

$$\int_C y^3dx + x^3dy = \int_{-2}^{1} [2(t^2-3)^3 + 8t^3(2t)]dt =$$

$$2\int_{-2}^{1} [t^6 - t^4 + 27t^2 - 27]dt = 2[t^7/7 - t^5/5 + 9t^3 - 27t]_{-2}^{1} =$$

$$2[1/7 - 1/5 + 9 - 27] - 2[-128/7 + 32/5 - 72 + 54]$$