

Math 2210-3. 1-st Midterm Test (sample): Solutions.

1. Find the parametric equation of the tangent line to the curve $P(t) = (e^{2t}, \frac{1}{t^2})$ at the point $P(1)$.

Solution. $P(1) = (e^2, 1)$. $\overrightarrow{OP}'(t) = (2e^{2t}, -\frac{2}{t^3})$ equals $(2e^2, -2)$ at $t = 1$. Hence the parametric equation of the tangent line is:

$$(e^2, 1) + t(2e^2, -2)$$

where t is the parameter.

2. Find the unit tangent vector $\overrightarrow{T}(0)$ and the curvature $k(0)$ of the curve

$$P(t) = (\sin^2(t), \sin(t))$$

at the point $P(0)$. Sketch the curve near $P(0)$, draw the vector $\overrightarrow{T}(0)$ with its initial point at $P(0)$.

Solution. If $t = 0$ then $P(t = 0) = (0, 0)$. Near $t = 0$ the function $x = \sin(t)$ is 1-1, hence we can make the change of variables $y = \sin(t)$ and find that near $t = 0$ our curve is given by the Cartesian equation $x = y^2$. We will think of x as the function of y . The curvature does not depend on the parameterization, thus the curvature of the curve $x = y^2$ at the point $(0, 0)$ is

$$k = \frac{|x''|}{(1 + (x')^2)^{3/2}} = \frac{2}{(1 + (2y)^2)^{3/2}} = \frac{2}{(1)^{3/2}} = 2.$$

Thus the curvature equals 2. The tangent vector at 0 is $\overrightarrow{OP}(0)' = (2 \cos(t) \sin(t), \cos(t))|_{t=0} = (0, 1)$. Clearly, this vector is a unit vector, hence $\overrightarrow{T}(0) = (0, 1)$. The sketch is below:

3. Find equation (in the form $Ax + By + Cz + D = 0$) of the plane Π (in the 3-space) which contains the point $P_0 = (0, 0, 1)$ and the line L :

$$\frac{x - 1}{2} = y + 1 = -z.$$

Solution. To find equation of a plane we need one point and a normal vector \overrightarrow{n} . The vector \overrightarrow{v} along the line L is $(2, 1, -1)$ (since L is given by a symmetric equation). This vector is parallel to the plane Π . To find another vector parallel to Π pick a point on L :

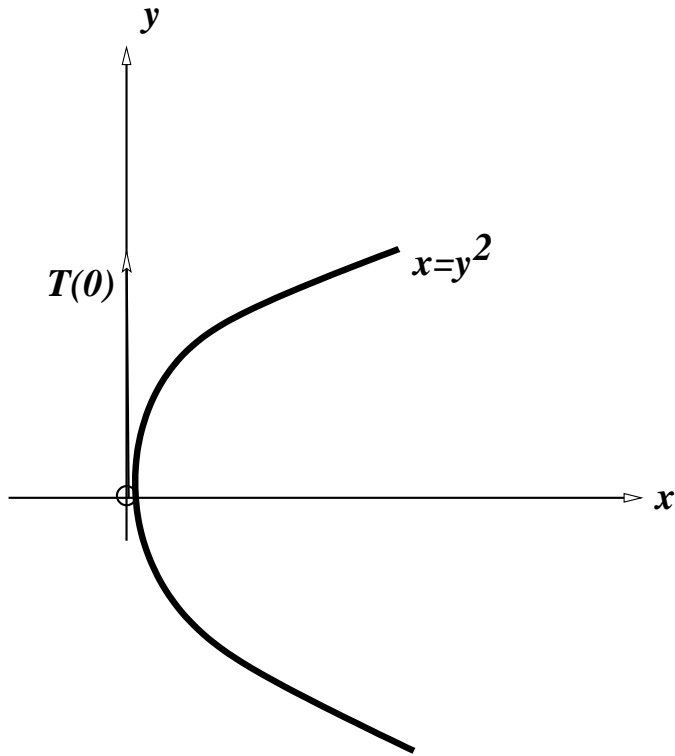
$$x = 1, y = -1, z = 0, Q_0 = (1, -1, 0).$$

Let $\overrightarrow{u} = \overrightarrow{Q_0P_0} = (1, 2, -1)$. Then

$$\overrightarrow{n} = \overrightarrow{v} \times \overrightarrow{u} = (1, 1, 3).$$

Hence Π is given by the equation:

$$x + y + 3z = (1, -1, 0) \cdot (1, 1, 3) = 1 - 1 = 0.$$



4. Compute the area of the parallelogram whose edges are the vectors (with the initial points at the origin) $\vec{u} = (0, 2, 2)$, $\vec{v} = (1, 0, 2)$.

Solution. The area is the same as the magnitude of the cross product of the two vectors:

$$\text{Area} = |\vec{u} \times \vec{v}| = \sqrt{4^2 + 2^2 + (-2)^2} = 2\sqrt{4 + 1 + 1} = 2\sqrt{6}.$$

5. Compute the distance from the point $P_0 = (1, 0)$ to the line L which passes through the points $Q_0 = (1, 1)$ and $R_0 = (2, 3)$.

Solution. Let $\vec{v} = \overrightarrow{Q_0R_0} = (1, 2)$. The line through the points Q_0 and R_0 is given by the symmetric equation:

$$\frac{x - 1}{1} = \frac{y - 1}{2}.$$

This means that $x - 1 = 2y - 1/2$, $2y - x + 1/2 = 0$. Hence the distance from $P_0 = (1, 0)$ to the line L equals

$$\frac{|2 \cdot 0 - 1 \cdot 1 + 1/2|}{\sqrt{2^2 + 1^2}} = \frac{1/2}{\sqrt{5}} = \frac{1}{2\sqrt{5}}.$$