Math 2210-3. Solutions for the 2-nd Test.

1. Does the following limit exist? Explain your solution.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+xy+2y^2}{x^2+xy+y^2}.$$

Solution. The limit does not exist. To see this consider first the limit along the x-axis, i.e., $y=0, x\to 0$:

$$\lim_{x\to 0} \frac{2x^2+0+0}{x^2+0+0} = 2.$$

Now consider the limit along the line $x=y$:

$$\lim_{x\to 0} \frac{2x^2+x^2+2x^2}{x^2+x^2+x^2} = 5/3.$$

These limits are different, hence the limit does not exist. □

2. Find and sketch the domain of the function $f(x,y)=\sqrt{x+y}-\sqrt{x-y}$.

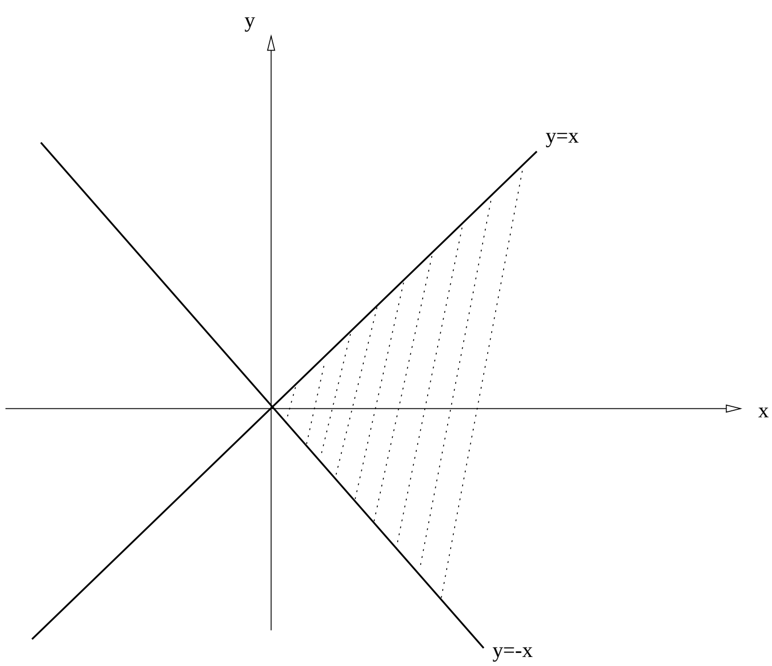

Figure 1: Problem 2.

Solution. The domain of the function f consists of (x,y) such that two inequalities are satisfied: $x+y\geq 0$ and $x-y\geq 0$. Equivalently, $y\geq -x$, $y\leq x$. Graphically we get a quarter of the plane, see Figure 1.

3. Find the unit vector in the direction of the fastest increase of the function $f(x,y)=x\sin(xy)+\cos^2(xy)$ at the point $P_0=(1,0)$.

Solution. $\nabla f=(f_x,f_y)$. By applying the chain rule we get:

$$f_x = \sin(xy)+x\cos(xy)y+2\cos(xy)(-\sin(xy))y =$$

$$= \sin(xy)+xy\cos(xy)-2y\cos(xy)\sin(xy).$$

$$f_y = x\cos(xy)x+2\cos(xy)(-\sin(xy))x = x^2\cos(xy)-2x\cos(xy)\sin(xy).$$

At the point $P_0 = (1,0)$ we have: $\nabla f(P_0) = (0,1)$. This vector has magnitude 1, hence the unit vector in the direction of the fastest increase of f at the point P_0 is the vector $(0,1)$. □

4. Find the critical points of the function

$$f(x,y) = x^3 + y^3 - xy$$

and determine which of them are local minima, maxima, saddle points.

Solution. $\nabla f = (f_x, f_y) = (3x^2 - y, 3y^2 - x)$. The function f is defined on the whole plane and is differentiable everywhere, hence its cricical points are the stationary points: $\nabla f(P) = (0,0)$:

$$3x^2 = y, 3y^2 = x,$$

hence $27x^4 = x$. This equation has two solutions: (1) $x = 0$ and (2) $x = 1/3$. In the first case $y = 0$, in the second case $y = 1/3$. Thus we have two stationary points: (1) $P_1 = (0,0)$, (2) $P_2 = (1/3, 1/3)$. We now compute the matrix of second derivatives:

$$d^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -1 \\ -1 & 6y \end{bmatrix}.$$

The determinant of this matrix at $P = P_1 = (0,0)$ equals $-1 < 0$, hence the critical point $P_1 = (0,0)$ is a saddle point.

The determinant of $d^2 f$ at the point $P = P_2 = (1/3, 1/3)$ equals $4 - 1 = 3 > 0$. On the other hand, $f_{xx}(P_2) = 2 > 0$. Thus $P_2 = (1/3, 1/3)$ is the point of local minimum.

5. Use Lagrange's method to find the maximal value of the function $f(x,y) = xy$ subject to the constraint $x + y = 2$.

Solution. First note that if $P = (x,y)$ is on the line $x + y = 2$ and belongs to the 2-nd or 4-th coordinate quadrant, then $xy < 0$. On the other hand, if $P = (x,y)$ is on the line $x + y = 2$ and belongs to the first coordinate quadrant then $xy > 0$. Thus maximizing the function f on the line $x + y = 2$ amounts to the same as maximizing this function on the interval of this line which is in the first coordinate quadrant. This interval is closed and bounded, hence the maximum exists. It remains to find the point of maximum. (Note that the function f equals zero in the end-points of the interval.) Let $g(x,y) = x + y - 2$. We know that the point of maximum is a critical point, hence it has to satisfy the Lagrange equations. Then

$$\nabla f(P) = \lambda \nabla g(P), g(P) = 0$$

at each critical point on the line $x + y = 2$. Thus:

$$(y, x) = (\lambda, \lambda), x + y = 2$$

$$x = y, 2x = x, x = 1, y = 1.$$

Thus the only critical point is $P_0 = (1,1)$. This point has to be the point of global maximum, $f(1,1) = 1$. Thus the maximal value of f on the line $x + y = 2$ is 1.

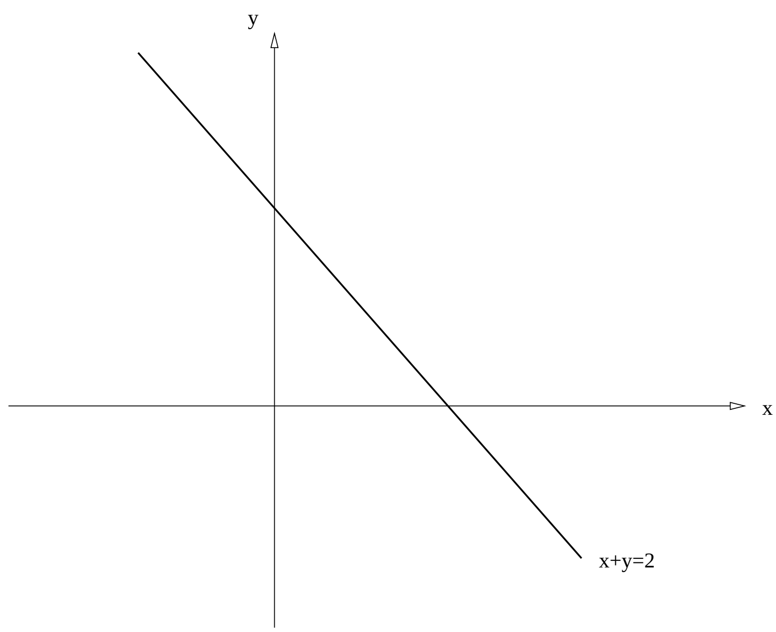

Figure 2: Problem 5.