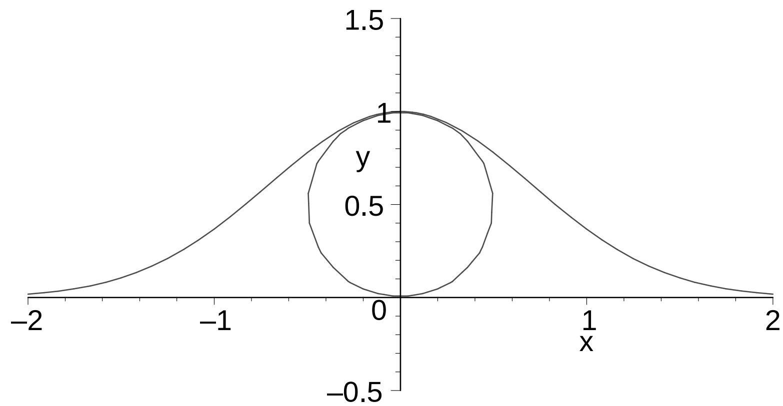
Math 2210-3. Solutions for the 1-st Test.

1. For which values of t are the following vectors orthogonal:

$$\vec{u} = (2t, t-1, 2) \text{ and } \vec{v} = (1, t, t).$$

Solution. Vectors are orthogonal when their dot product is zero. $0 = \vec{u} \cdot \vec{v} = (2t, t-1, 2) \cdot (1, t, t) = 2t + t^2 - t + 2t = t^2 + 3t$. The equation $t^2 + 3t = 0$ has two solutions: $t = 0$ and $t = -3$. Hence the vectors are orthogonal when $t = 0$, $t = -3$. □

2. Sketch the curve $y = e^{-x^2}$. Compute the curvature of this curve at the point $P_0 = (0, 1)$. Bonus: on the same picture sketch the circle of curvature at the point P_0.



Solution. $y' = -2xe^{-x^2}$, $y'' = -2e^{-x^2} + 4x^2e^{-x^2}$. The intersections with the coordinate lines: $(0, 1)$ (with the y-axis), $y(x) > 0$ for all x, hence no intersection with the x-axis. The function $y(x)$ is increasing for $x > 0$ (since $y' > 0$ for $x < 0$) and is decreasing for $x > 0$. As $x \to \pm\infty$, the function converges to zero. Hence the x-axis is the horizontal asymptota in the both directions. The sketch is above.

At $x = 0$ we get: $y'(0) = 0$, $y''(0) = -2$. Hence the curvature equals:

$$k(0) = \frac{|y''|}{(1 + (y')^2)^{3/2}} = \frac{|-2|}{(1+0)^{3/2}} = 2.$$

Hence the curvature at P_0 equals 2.

Bonus: To sketch the curvature circle recall that the radius of the curvature equals $1/k = 1/2$. The center of the curvature circle is on the head of the vector of $\vec{K}(0)/k^2$ (whose tail is at P_0). The vector of curvature $\vec{K}(0)$ is orthogonal to the curve, has the magnitude $k = 2$ and is directed towards the convex side of the curve. Hence $|\vec{K}(0)/k^2| = 1/2$, and the center of the circle of curvature is at the point $(0, 1/2)$.

3. Find equation of the plane which contains the points $A = (1, 1, 1)$, $B = (0, 1, 0)$ and $C = (3, 5, 4)$.

Solution. Let $\vec{u} = \overrightarrow{BA} = (1, 0, 1)$, $\vec{v} = \overrightarrow{BC} = (3, 4, 4)$. Then the normal vector to the plane is given by the cross-product:

$$\vec{n} = \vec{u} \times \vec{v} = (-4, -1, 4).$$

The equation of the plane is $\vec{\mathbf{x}} \cdot \vec{n} = \overrightarrow{OB} \cdot \vec{n}$:

$$-4x - y + 4z = (0, 1, 0) \cdot (-4, -1, 4) = -1.$$

Hence the equation of the plane is: $-4x - y + 4z = -1$ (or $4x + y - 4z = 1$). $\square$

4. Find the symmetric equation of the line L which contains the point $P_0 = (2, 3, 1)$ and which is perpendicular to the plane $x + y = 0$.

Solution. The normal vector to this plane is $\overrightarrow{v} = (1, 1, 0)$. Since the line L is perpendicular to the plane, this line is parallel to the vector $\overrightarrow{v}$. Thus we know a point $P_0 = (2, 3, 1)$ on the line L and the vector $\overrightarrow{v} = (1, 1, 0) = (a, b, c)$ along this line. In general, if we are given such data, the symmetric equation has the form:

$$\frac{x-2}{a} = \frac{y-3}{b} = \frac{z-1}{c}.$$

However in our case $c = 0$ and we cannot divide by zero! On the other hand, since $c = 0$, each point on the line satisfies $z - 1 = 0$. Hence we get the equations:

$$\frac{x-2}{1} = \frac{y-3}{1}, \quad z = 1,$$

or

$$x - 2 = y - 3, \quad z = 1.$$

5. Sketch the vector $\overrightarrow{u} + \overrightarrow{v} - \overrightarrow{2w}$.

Solution. First sketch $\overrightarrow{u} + \overrightarrow{v}$. Then compute $-\overrightarrow{2w}$ by reversing the direction of $\overrightarrow{w}$ and expanding it by 2. Finally use the triangle rule to add the vector $-\overrightarrow{2w}$ to $\overrightarrow{u} + \overrightarrow{v}$.