# TOPOLOGICAL ASPECTS OF KLEINIAN GROUPS IN SEVARAL DIMENSIONS

MICHAEL KAPOVICH

## 1. Introduction

The goal of this survey is to give an overview (from a topologist's prospective) of the theory of Kleinian groups in higher dimensions. The survey grew out of a series of lectures which I have given in the University of Maryland in Fall of 1991 and which I have decided to update and publish now, in the proceedings of the 3-rd Ahlfors-Bers Colloquium. This survey was greatly influenced by the work of Lars Ahlfors, even the title was partly borrowed from his lecture notes [1].

Kleinian groups in higher dimensions is a vast subject; in this paper I organize them according to the topology of their limit sets, trying to contrast and compare them with the Kleinian subgroups of $PSL(2,\mathbb{C})$. The groups with zero-dimensional limit sets are relatively easy to understand (section 3). In the case of convex-cocompact groups with 1-dimensional limit sets, at least the topology of the limit sets is understood (see section 4), although their group-theoretic structure is a mystery. We know very little about Kleinian groups with higher-dimensional limit sets, thus I have restricted the discussion to Kleinian groups whose limit sets are topological spheres (section 5). I then discuss Ahlfors finiteness theorem and its failure in higher dimensions (section 6). Next I consider the representation varieties of Kleinian groups and topological and geometrical constrains on Kleinian groups in higher dimensions. Lastly, I discuss generalizations of Kleinian groups: uniformly quasiconformal groups, fundamental groups of negatively curved manifolds and the convergence groups.

## 2. Definitions and notation

We let $\mathbb{B}^{n+1}$ denote the closed ball $\mathbb{H}^{n+1}\cup\mathbb{S}^n$, its boundary is identified via the stereographic projection with $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$. Let $\mathbf{Mob}(\mathbb{S}^n)$ denote the group of all orientation-preserving Moebius transformations of the $n$-sphere $\mathbb{S}^n$; this group is isomorphic to the connected component of the identity in the Lorentz group $SO(n+1,1)$. Discrete subgroups $\Gamma\subset\mathbf{Mob}(\mathbb{S}^n)$ are called *Kleinian groups*; note that we do not require nonempty domain of discontinuity in $\mathbb{S}^n$.

For a group $\Gamma\subset\mathbf{Mob}(\mathbb{S}^n)$ the *discontinuity set* $\Omega(\Gamma)$ is the largest open subset in $\mathbb{S}^n$ where $G$ acts properly discontinuously. Its complement $\mathbb{S}^n\setminus\Omega(\Gamma)$ is the *limit set* $\Lambda(\Gamma)$ of the group $\Gamma$. We will use the notation $M^n(\Gamma)$ for the quotient $\Omega(\Gamma)/\Gamma$ and $\bar{M}^{n+1}(\Gamma)$ for the $n+1$-dimensional quotient $(\mathbb{H}^{n+1}\cup\Omega(G))/G$. A Kleinian group is called *elementary* if its limit set is finite.

*Date*: May 12, 2002.

Supported by NSF grant DMS-8902619 at the University of Maryland and DMS-99-71404 at University of Utah.

If $\Gamma$ is a Kleinian group then $\Gamma$-conjugacy classes of maximal parabolic subgroups of $G$ are called *cusps* of $\Gamma$. For a Kleinian subgroup $\Gamma \subset \mathrm{Mob}(\mathbb{S}^n)$ let $Hull(\Lambda(\Gamma))$ denote the convex hull in $\mathbb{H}^{n+1}$ of $\Lambda(\Gamma)$, i.e. the smallest closed convex subset in $\mathbb{H}^{n+1}$ whose *ideal boundary* is $\Lambda(\Gamma)$.

For $\epsilon > 0$ consider the $\epsilon$-neighborhood $Hull(\Lambda(\Gamma))_\epsilon$ of $Hull(\Lambda(\Gamma))$ in $\mathbb{H}^{n+1}$. Since $Hull(\Lambda(\Gamma))_\epsilon$ is $G$-invariant, we can form the quotient $Hull(\Lambda(\Gamma))_\epsilon/\Gamma$. Recall that the group $\Gamma$ is called *geometrically finite* if:

(1) $\Gamma$ is finitely generated and

(2) $vol(Hull(\Lambda(\Gamma))_\epsilon)/\Gamma < \infty$.

Note that E. Hamilton [45] constructed an example of a Kleinian group $\Gamma \subset \mathrm{Mob}(\mathbb{S}^3)$ for which (2) holds but (1) fails. There are other definitions of geometric finiteness (see [18]). A discrete subgroup $\Gamma \subset \mathrm{Mob}(\mathbb{S}^n)$ is called a *lattice* if $\mathbb{H}^{n+1}/\Gamma$ has finite volume; in this case $\Gamma$ is necessarily geometrically finite.

**Assumption 2.1.** *From now on we will assume that all Kleinian groups are finitely generated and torsion free, unless we state otherwise. The second part of this assumption is not very restrictive because of Selberg lemma.*

One can characterize geometrically finite groups in terms of their action on the limit set:

**Theorem 2.2.** *(A. Beardon, B. Maskit* [7], *B. Bowditch* [18]*) $G$ is geometrically finite iff each limit point $x \in \Lambda(G)$ is either a* conical limit point *or a* bounded parabolic fixed point.

Recall that $x \in \Lambda(G)$ is a *conical limit point* if (assuming $G$ is nonelementary) for every $y \in \Lambda(G) - \{x\}$ there exists a sequence $g_k \in G$ such that $\lim_{k\to\infty} g_k(z) = y$ for every $z \in \Lambda(G) - \{x\}$. If $G$ is elementary, then $x$ is a conical limit point if it is fixed by a hyperbolic element of $G$.

The point $x \in \Lambda(G)$ is a *bounded parabolic fixed point* if $x$ is a fixed point of a parabolic subgroup $P \subset G$ and $(\Lambda(G) - \{x\})/P$ is compact.

*Remark* 2.3. C. Bishop proved in [13] that the weaker assumption, *each limit point is either a conical limit point or a parabolic fixed point*, still implies geometric finiteness.

If $G$ has no parabolic elements then geometric finiteness is equivalent to the assumption that the manifold $\bar{M}^{n+1}(\Gamma)$, or, equivalently, $Hull(\Lambda(\Gamma))_\epsilon/\Gamma$, is compact. In this case the group $\Gamma$ is called *convex-cocompact.* In the presence of parabolic elements one has to remove from $\mathbb{H}^{n+1} \cup \Omega(\Gamma)$ a $G$-invariant collection $U$ of disjoint *cuspidal neighborhoods* of parabolic fixed points of $\Gamma$. Then $\Gamma$ is geometrically finite provided that

$$(\mathbb{H}^{n+1} \cup \Omega(\Gamma) \setminus U)/\Gamma$$

is compact. If $n = 2$ then the classical definition of geometrical finiteness requires existence of a convex finitely-sided fundamental polyhedron for the action of $G$ in $\mathbb{H}^3$, however in the higher dimensions such requirement is questionable.

In this paper we shall also discuss an object closely related to the theory of Kleinian groups, namely *Moebius structures.* When $\dim(M) \geq 3$, a *Moebius* (or *flat conformal*) structure $K$ on a manifold $M$ is a conformal class of conformally-Euclidean Riemannian metrics. From the topological point of view, $K$ is a maximal *Moebius atlas* on $M$, i.e. an atlas class with Moebius transition maps. Thus, the standard flat conformal structure on $\mathbb{S}^n$ projects to a Moebius structure $K_G$ on the manifold $M^n(G) = \Omega(G)/G$ provided that $G$ acts freely on its discontinuity domain. The structures of such kind are called *uniformizable.*

**Sources of Kleinian groups**:

(a) Arithmetic groups and their subgroups.

(b) Poincare fundamental polyhedron theorem.

(c) Klein–Maskit Combination Theorems.

(d) Small deformations of a given Kleinian group.

(e) Limits of sequences of Kleinian groups.

The source (b), in principle, is the most general. The Poincare fundamental polyhedron theorem (see [74], [86]) tells us that given a polyhedron $\Phi$ in $\mathbb{H}^{n+1}$ and a family of isometries $g_1, g_2, ..., g_k, ...$ pairing the faces of $\Phi$, under certain condition on this "data" the group $G$ generated by $g_1, g_2, ..., g_k, ...$ is discrete and $\Phi$ is a fundamental domain for the action of the group $G$ on $\mathbb{H}^{n+1}$. Each Kleinian group has a fundamental polyhedron (for example, Dirichlet fundamental domain). However, in practice, it is not always easy to use this theorem, especially if $\Phi$ has infinitely many faces. Another problem is that almost always the construction of $\Phi$ is rather art than science.

The Klein–Maskit Combination Theorems (see [74]) provide conditions that guarantee that a group $G$ generated by two Kleinian subgroups $G_1, G_2 \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ is again Kleinian and splits as an amalgam of $G_1$ and $G_2$; Combination Theorems also show that the quotient space $M^n(G)$ of the group $G$ is obtained from the quotients $M^n(G_1)$ , $M^n(G_2)$ via some "cut-and-paste" operation.

A subgroup $\Gamma \subset SO(n+1, 1)$ is called *arithmetic* if there is an embedding

$$\iota : SO(n+1, 1) \hookrightarrow GL(N, \mathbb{R})$$

such that the image $\iota(\Gamma)$ is commensurable with the intersection $\iota(SO(n+1, 1)) \cap GL(N, \mathbb{Z})$. Recall that two groups $A$ and $B$ are called *commensurable* if $A \cap B$ has finite index in both $A$ and $B$.

A beautiful example of application of (a) and (c) is the construction of Gromov and Piatetski-Shapiro [40] of *non-arithmetic* lattices in $\mathbf{M}\mathrm{ob}(\mathbb{S}^n)$. Starting with two arithmetic groups $\Gamma_j$ ($j = 1, 2$) they first "cut these groups in half", take "one half" $G_j \subset \Gamma_j$ of each, and then combine $G_j$'s via Maskit Combination. The construction of Kleinian groups in [41] is an application of (a), (c) and (d).

Currently the most sophisticated construction of Kleinian groups is given by Thurston's hyperbolization theorem, however still it is essentially a combination (of course, a very complicated one!) of (c), (d) and (e).

Thurston's hyperbolic Dehn surgery theorem is an example of (e).

There is potentially the sixth source of Kleinian groups in higher dimensions: monodromy of differential equations; however, to the best of my knowledge, it was used only to construct discrete subgroups of $PU(n, 1)$ (i.e. discrete isometry groups of complex-hyperbolic space) by Deligne and Mostow, see [29].

**Notions of equivalence for Kleinian groups:**

(0) $G_1$ is isomorphic to $G_2$.

(1) **Dynamical equivalence**: there is a homeomorphism $f : \Lambda(G_1) \to \Lambda(G_2)$ such that $fG_1f^{-1} = G_2$; i.e., the groups $G_1$ and $G_2$ have the same topological dynamics on their limit sets. Thus, $G_1$ is geometrically finite iff $G_2$ is, since geometric finiteness can be stated in terms of topological dynamics of a group on its limit set.

(2) **Topological conjugation**: for the pair of Kleinian groups $G_1, G_2 \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $fG_1f^{-1} = G_2$. (One can relax this by assuming that $f$ is defined only on the domain of discontinuity.)

(3) **Quasiconformal conjugation**: in (2) one can find a quasiconformal homeomorphism.

(4) **Topological isotopy**: in (2) there exists a continuous family of homeomorphisms $h_t : \mathbb{S}^n \to \mathbb{S}^n$ such that: $h_0 = id$; $\forall t, h_tG_1h_t^{-1} \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ ; $h_1G_1h_1^{-1} = G_2$.

(5) **Quasiconformal isotopy**: in (4) all homeomorphisms are quasiconformal.

(6) **Conjugation:** there is $f \in \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ such that $fG_1f^{-1} = G_2$.

Suppose that both groups $G_j$ are geometrically finite and $\varphi : G_1 \to G_2$ is an isomorphism which preserves the *type* of elements (i.e. the images of hyperbolic elements are hyperbolic, etc.).

**Theorem 2.4.** *(P. Tukia* [97]*). Under the about assumptions the isomorphism $\varphi$ can be realized by an equivalence (1), i.e. $fgf^{-1} = \varphi(g)$ for all $g \in G_1$.*

**Question 2.5. (Quasiconformal vs. topological.)** Suppose that two Kleinian groups $G_1, G_2 \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ are topologically conjugate by a homeomorphism $f$ (which is defined either on $\mathbb{S}^n$ or on the limit set or on $\Omega(G_1)$). Does it imply that they can be conjugate via a quasiconformal homeomorphism (of $\mathbb{S}^n$, of the limit sets or of the discontinuity domains resp.)?

If $n = 1$ then the answer is "yes". If $f$ is defined on $\Omega(G_1)$ then the answer is positive provided that $n \neq 4$ and $M(G_1)$ is compact (this is a special case of a theorem of D. Sullivan [92]). If $n = 4$ and $M(G_1)$ is compact then the situation is unclear but one probably should expect the negative answer (see [30]). If $n = 2$ then the answer is again positive for geometrically finite groups (A. Marden [70]). Moreover, it follows that topological conjugation implies existence of a quasiconformal isotopy on $\mathbb{S}^2$. For geometrically infinite groups the question is open. In the case of freely indecomposable Kleinian groups the positive answer follows from the recent proof of Thurston's Ending Lamination Conjecture by Brock, Canary and Minsky.

However if we consider geometrically finite Kleinian groups in $\mathbf{M}\mathrm{ob}(\mathbb{S}^3)$ which contain parabolic elements then the answer is negative since two parabolic transformations are always conjugate topologically, but not always quasiconformally.

The following is open in higher dimensions even for convex-cocompact groups:

**Question 2.6.** (1) $\Rightarrow$ (4), (3) $\Rightarrow$ (5)?

For an abstract group $\Gamma$ let $cd(\Gamma)$ denote the *cohomological dimension* of $\Gamma$ (over $\mathbb{Z}$). Intuitively, the cohomological dimension of $\Gamma$ is the smallest dimension of an acyclic (over $\mathbb{Z}$) manifold $X$ on which $\Gamma$ can act freely properly discontinuously. The actual definition requires a bit of homological algebra (instead of working with acyclic manifolds one has to work with projective resolutions), see [22].

**Example 2.7.** 1. Suppose that $\Gamma$ is the fundamental group of a close aspherical $n$-manifold. Then $cd(\Gamma) = n$.

2. $cd(\Gamma) = 1$ iff $\Gamma$ is free.

## 3. Groups with zero-dimensional limit sets

Suppose that $G$ is non-elementary and $\Lambda(G)$ has topological dimension 0. It follows that $\Lambda(G)$ is totally disconnected and thus $\Lambda(G)$ homeomorphic to a Cantor set $K \subset [0, 1]$.

**Question 3.1.** How is $\Lambda(G)$ embedded in $\mathbb{S}^n$?

If $n = 2$ then for every nonempty totally disconnected perfect compact subset (such sets are called *discontinuums*) $C \subset \mathbb{S}^2$, there exists a homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ which maps $C$ onto a Cantor set $K \subset [0, 1]$. Moreover, if $C = \Lambda(G)$ and $G$ contains no parabolic elements then $G$ is free and one can assume that $f : \mathbb{S}^2 \to \mathbb{S}^2$ is quasiconformal, so that $fGf^{-1} = G'$ is a Kleinian group whose limit set is a Cantor subset of $[0, 1]$.

If $n = 3$ then there are examples of *wild* topological embeddings $\iota : K \hookrightarrow \mathbb{S}^3$, which means that there is no ambient homeomorphism which maps $\iota(K) = C$ onto a subset of the real line. (An embedding $\iota$ which is not wild is called *tame*.)

The standard example of a wild embedding $\iota$ is given by the *Antoine discontinuum* in $\mathbb{S}^3$, whose complement is not simply-connected. There are slightly more sophisticated examples when $\pi_1(\mathbb{S}^3 - C) = 1$ but $C$ is still wild[1].

Here is a couple of examples of Kleinian groups whose limit sets are totally disconnected.

**Example 3.2.** (A **Schottky group**.) Let $n, k \geq 2$. Suppose that we are given a collection of disjoint closed topological $n$-disks with smooth boundary

$$B_1, B_2, \dots B_k, B'_1, \dots B'_k \subset \mathbb{S}^n$$

and Moebius transformations $g_j : \mathrm{int}(B_j) \to \mathrm{ext}(B'_k)$. Then

$$\Phi := \mathbb{S}^n - \bigcup_{j=1}^{n} (B_j \cup \mathrm{int}(B'_k))$$

is a fundamental domain for the group $G$ generated by $g_1, \dots, g_k$. The group $G$ is isomorphic to a free group of rank $k$, and the limit set of $G$ is a *tame* discontinuum in $\mathbb{S}^n$. Every two such groups are topologically conjugate (provided that the rank is the same).

The quotient manifold $M^n(G)$ for such group is homeomorphic to the connected sum of $k$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$: it is obtained from $\Phi$ by identifying the boundary components via the generators $g_i$'s. This homeomorphism is clear in the dimensions 2 and 3 but requires more complicated arguments in higher dimensions (including the

[1] One can find $C$ for which the *local fundamental group* of $\mathbb{S}^3 - C$ near a point $x \in C$ is nontrivial; such $C$ would have to be wild.

topological Shoenflies theorem). Each Schottky group is convex-cocompact. Actually, this is not completely obvious since, a priori, the fundamental domain $\Phi$ does not extend to a similar fundamental domain in $\mathbb{B}^{n+1}$. The fundamental domain clearly extends if we assume that all topological balls $B_j, B_i'$ are *round*. Such groups are called *classical Schottky groups*; these groups are obviously convex-cocompact. If $G, G'$ are Schottky groups (with fundamental domains $\Phi, \Phi'$) of the same rank, one can easily see that they are *dynamically equivalent*. To construct a homeomorphism $f : \Lambda(G) \to \Lambda(G')$ start with an isomorphism $\phi : G_1 \to G_2$, pick a point $x \in \Phi$, send it to $x' \in \Phi'$ and then define a homeomorphism between the orbits:

$$h : Gx \to G'x', \quad h(gx) = \phi(g)(x').$$

It is clear that the map $h$ and its inverse are uniformly continuous (since the limit sets are totally disconnected). Thus the map $h$ extends to a homeomorphism $f : \Lambda(G) \to \Lambda(G')$ which is $\phi$-equivariant:

$$f \circ g = \phi(g) \circ f, \forall g \in G.$$

Therefore $G$ and $G'$ are dynamically equivalent and hence one is convex-cocompact iff the other is. Hence each Schottky group is convex-cocompact. As an alternative argument notice that for each Schottky group $G$, the map

$$j : H_n(M^n(G)) \to H_n(\bar{M}^{n+1}(G))$$

is trivial for each Schottky group $G$ (for instance, because it is true for the classical Schottky groups). If $G$ were not convex-cocompact, then $\bar{M}^{n+1}(G) = (\mathbb{H}^{n+1} \cup \Omega(G))/G$ would be noncompact, this would mean that $j$ is injective, which is a contradiction.

Note that the above example could be described via *Klein combination theorem*: we start with the elementary Kleinian groups $G_i = \langle g_i \rangle$; each has a fundamental domain $\Phi_i$ which is the exterior of $B_i \cup B_i'$, $i = 1, ..., k$. Thus the intersection of the complements $\Phi_i^c \cap \Phi_j^c$ is empty unless $i = j$. Suppose that $H \subset \mathrm{Mob}(\mathbb{S}^n)$ is an elementary group which fixes the point $\infty \in \overline{\mathbb{R}^n}$; we say that a fundamental domain $\Psi$ of $H$ is *standard* if it is either a convex Euclidean polyhedron with finitely many faces (if $H$ consists of parabolic elements) or if it is bounded by two disjoint round spheres. If $H' \subset \mathrm{Mob}(\mathbb{S}^n)$ is elementary, then we call a fundamental domain $\Psi'$ for $H$ *topologically standard* if there is an elementary group $H$ (fixing $\infty$), standard fundamental domain $\Psi$ for $H$ and a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which conjugates $H$ to $H'$ and sends $\Psi$ onto $\Psi'$.

**Example 3.3.** (**Schottky-type groups.**) Start with a collection of elementary Kleinian groups $G_i$, $i = 1, ..., k$. Let $\Phi_i \subset \mathbb{S}^n$ be topologically standard fundamental domains for such groups. For instance, if $G_i$ is isomorphic to $\mathbb{Z}^n$ then we can take $\Phi_i$ diffeomorphic to a cube in $\mathbb{R}^n$. Assume again that $\Phi_i^c \cap \Phi_j^c$ is empty unless $i = j$. (This could be easily arranged by conjugating $G_i$'s by appropriate Moebius transformations.) Then

$$\Phi := \Phi_1 \cap ... \cap \Phi_k$$

is a fundamental domain for the group $G$ which is generated by $G_1, ..., G_k$. As an abstract group, $G$ is isomorphic to the free product $G_1 * ... * G_k$. The group $G$ is Kleinian and its limit set is totally disconnected. Groups $G$ obtained in this fashion are called *Schottky-type groups.* Arguing analogously to the previous example one can show that such groups are always geometrically finite.

According to B. Maskit [73], Schottky groups acting on $\mathbb{S}^2$ can be characterized by the following nice

**Theorem 3.4.** *A Kleinian subgroup of* $\mathbf{Mob}(\mathbb{S}^2)$ *is a Schottky group iff* $G$ *is free, has nonempty domain of discontinuity in* $\mathbb{S}^2$ *and* $G$ *consists only of hyperbolic transformations.*

This result was generalized by N. Gusevskii and N. Zindinova [44] as follows:

**Theorem 3.5.** *Let* $G$ *be a Kleinian subgroup of* $\mathrm{Mob}(S^2)$ *which has nonempty domain of discontinuity in* $\mathbb{S}^2$ *and is isomorphic to a Schottky-type group via a type-preserving isomorphism. Then* $G$ *is a Schottky-type group itself.*

Both theorems are trivial under the assumption that $G$ is geometrically finite, the key point here is that (in dimension 2) one can prove geometric finiteness under the mild assumptions above.

If $G$ is a Kleinian subgroup of $\mathbf{M}\mathrm{ob}(S^3)$, then the above results are not longer true, moreover $G$ can easily be geometrically infinite. For instance, take a free (finitely generated!) purely hyperbolic discrete subgroup of $PSL(2,\mathbb{C})$, whose limit set is the whole sphere (existence of such groups first was established by V. Chuckrow [25]). The extension of this group to $\mathbb{S}^3$ has nonempty domain of discontinuity, but is not geometrically finite (otherwise its limit set would be totally disconnected!).

**Theorem 3.6.** *(R. Kulkarni* [68]*.) Suppose that a Kleinian group* $G \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ *has totally disconnected limit set. Then* $G$ *is isomorphic to a Schottky type group.*

**Theorem 3.7.** *(N. Gusevskii,* [42]*.) Suppose that* $G$ *is as in the above theorem. Then* $G$ *admits a fundamental domain* $\Phi$ *of the same shape as in example 3.3, only the fundamental domains* $\Phi_i$ *for* $G_i$*'s are not required to be topologically standard.*

**Corollary 3.8.** *Theorem 3.7 implies that each Kleinian group with totally disconnected limit set is geometrically finite.*

*Proof.* Repeat the arguments which we used to establish geometric finiteness of Schottky groups. □

The proof of theorem 3.7 is based on the following

**Theorem 3.9.** *(M. Brin,* [21]*.) Let* $\tilde{M}$ *be an oriented manifold of dimension* $> 2$ *and* $G \curvearrowright \tilde{M}$ *is a a properly discontinuous free action. Assume also that every closed 2-sided hypersurface in* $\tilde{M}$ *separates* $\tilde{M}$*;*

*Then, for every oriented compact hypersurface* $\Sigma$ *in* $\tilde{M}$ *and an open neighborhood* $U$ *of* $G \cdot \Sigma$ *there exists a compact connected oriented hypersurface* $\Sigma^* \subset U$ *such that for every* $g \in G$ *either* $g\Sigma^* \cap \Sigma^* = \emptyset$ *or* $g\Sigma^* = \Sigma^*$*.*

This result allows one to split (inductively) the Kleinian group $G$ as a free product in a "geometric fashion": Start with a compact hypersurface in $\Omega(G)$ which separates components of $\Lambda(G)$. Find $\Sigma^*$ which still separates. Then cut open the manifold $M^n(G)$ along the projection of $\Sigma^*$. This decomposition yields a free product decomposition $G = G_1 * G_2$ so that $G$ is a Klein combination of the groups $G_1, G_2$. Continue inductively. Finite generation of $G$ implies that the decomposition process will terminate and the terminal groups must be elementary. Note that if all $\Sigma^*$ were spheres then this decomposition would be Schottky-type.

**Theorem 3.10.** *(M. Bestvina, D. Cooper* [11].*) There exists a finitely generated Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^3)$ which contains parabolic elements and has totally disconnected limit set with non-simply connected complement in $\mathbb{S}^3$.*

The proof that $\pi_1(\Omega(G)) \neq 1$ presented in [11], was incomplete; however the gap was filled several years later by S. Matsumoto [75, 76]:

**Theorem 3.11.** *(S. Matsumoto* [75, 76]*, see also* [43].*) There are examples of Kleinian groups $G$ in $\mathrm{Mob}(\mathbb{S}^3)$ without parabolic elements whose limit sets are totally disconnected and wild.*

*Remark* 3.12. It is easy to see that if $G \subset \mathrm{Mob}(\mathbb{S}^3)$ has simply-connected $\Omega(G)$ and zero-dimensional $\Lambda(G)$, then $G$ is a Schottky-type. In particular, the limit set of $G$ is tame. To prove this use the sphere theorem for 3-manifolds.

## 4. Groups with one-dimensional limit sets

Let $\dim(\cdot)$ stands for the topological (covering) dimension of a topological space. We recall the construction of two standard examples of 1-dimensional compact sets.

**The Sierpinsky carpet $\mathcal{S}$.** Start with the unit square $S = I \times I$. Subdivide this square into 9 squares of the size $\frac{1}{3} \times \frac{1}{3}$ and then remove from $S$ the open middle square $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. Repeat this for each of the remaining $\frac{1}{3} \times \frac{1}{3}$ subsquares in $S$ and continue inductively. After removing a countable collection of open squares we are left with a compact subset $\mathcal{S} \subset \mathbb{R}^2$, called the *Sierpinsky carpet.*

**The Menger curve $\mathcal{M}$.** Start with the unit cube $Q = I \times I \times I$. Each face $F_i$ of $Q$ contains a copy of the Sierpinsky carpet $\mathcal{S}_i$. Let $p_i : Q \to F_i$ denote the orthogonal projection. Finally, let

$$\mathcal{M} := \bigcap_i p_i^{-1}(\mathcal{S}_i).$$

**Example 4.1.** There exists a convex-cocompact subgroup $G \subset \mathrm{Mob}(\mathbb{S}^2)$ whose limit set is homeomorphic to the Sierpinsky carpet $\mathcal{S}$.

To construct such example start with a compact hyperbolic manifold $M^3$ with nonempty totally-geodesic boundary. Then the limit set for the action of $G = \pi_1(M)$ on $\mathbb{S}^2$ is homeomorphic to the Sierpinsky carpet. To see this note that the convex hull $Hull(\Lambda(G))$ of the limit set of $G$ in $\mathbb{H}^3$ is obtained by removing a countable collection of disjoint open half-spaces $H_j \subset \mathbb{H}^3$ from $\mathbb{H}^3$. The ideal boundary of each $H_j$ is the open round disk $D_j \subset \mathbb{S}^2$. Thus $\Lambda(G) = \mathbb{S}^2 \setminus \bigcup_j \mathrm{int}(D_j)$. Clearly, $D_j \cap D_i = \emptyset$, unless $i = j$; since $\Lambda(G)$ has empty interior, we conclude that $\Lambda(G)$ is homeomorphic to $\mathcal{S}$, see [26].

**Example 4.2.** (M. Bourdon, [17]) There exists a convex-cocompact subgroup $G \subset \mathrm{Mob}(\mathbb{S}^3)$ whose limit set is homeomorphic to the Menger curve $\mathcal{M}$.

**Theorem 4.3.** *(M. Kapovich, B. Kleiner, [59]) Suppose that $G \subset \mathrm{Mob}(\mathbb{S}^n)$ is a (torsion-free) nonelementary convex-cocompact group such that: (a) G does not split as a free product, (b) G does not split as an amalgam over $\mathbb{Z}$, (c)* $\dim(\Lambda(G)) = 1$. *Then $\Lambda(G)$ is either homeomorphic to the Sierpinsky carpet or to the Menger curve.*

**Conjecture 4.4.** *(See [59].) If $\Gamma \subset \mathrm{Mob}(\mathbb{S}^n)$ is a torsion-free convex-cocompact Kleinian group whose limit set is homeomorphic to the Sierpinsky carpet, then $\Gamma$ is isomorphic to a convex-cocompact subgroup in* $\mathrm{Mob}(\mathbb{S}^2)$.

In [59] this conjecture was reduced to Cannon's conjecture (see Conjecture 8.5).

## 5. Groups whose limit sets are topological spheres

**Definition 5.1.** A Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^n)$ is called *i-fuchsian*[2] if the limit set of $G$ is a round $i$-dimensional sphere in $\mathbb{S}^n$.

To construct examples of $i$-fuchsian groups start with a lattice $\Gamma \subset \mathrm{Mob}(\mathbb{S}^i)$. The limit set of $\Gamma$ is the round $i$-sphere. Extension of $\Gamma$ to $\mathbb{S}^n$ gives examples of $i$-fuchsian subgroups of $\mathrm{Mob}(\mathbb{S}^n)$. One can modify this construction as follows. Note that the stabilizer of $\mathbb{S}^i$ in $\mathrm{Mob}(\mathbb{S}^n)$ contains $\mathrm{Mob}(\mathbb{S}^i) \times SO(n-i)$ as a subgroup of index 2. Suppose that $\phi : G \to SO(n-i)$ is a homomorphism. Then the image of

$$id_G \times \phi : G \to \mathrm{Mob}(\mathbb{S}^i) \times SO(n-i) \subset \mathrm{Mob}(\mathbb{S}^n)$$

is also an $i$-fuchsian group.

**Definition 5.2.** A Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^n)$ is called *i-quasifuchsian* if the limit set of $G$ is a topological $i$-dimensional sphere.

We will refer to the number $n - i$ as the *codimension* of a (quasi)fuchsian group $G$.

**Example 5.3.** Suppose that $G$ is an $i$-fuchsian subgroup of $\mathrm{Mob}(\mathbb{S}^n)$ and $G' \subset \mathrm{Mob}(\mathbb{S}^n)$ is another group which is topologically conjugate to $G$ (with the homeomorphism $f$ defined on the entire sphere). Then $G'$ is $i$-quasifuchsian. However there are $i$-quasifuchsian groups (if $n \geq 3$) which cannot be obtained in this fashion.

The (quasiconformal) homeomorphisms $f$ as in the previous example exist in abundance if $i = 1, n = 2$, thanks to solvability of the Beltrami equation. If $i \geq 2$, the situation is very different and it is not so easy to construct nontrivial examples of $i$-quasifuchsian groups which are not fuchsian.

**Theorem 5.4.** *Each convex-cocompact i-quasifuchsian group is a* Poincare duality group *of dimension $i+1$, see* [12].

Instead of giving a precise definition of Poincare duality groups (see [22]) I only note here that the fundamental group of each closed orientable aspherical manifold $M^k$ is a $k$-dimensional Poincare duality group. There are Poincare duality groups which are not like this but the only known examples are not finitely-presentable.

**Question 5.5.** Is it true that each convex-cocompact quasifuchsian group is isomorphic to the fundamental group of a closed aspherical manifold?

[2]Our definition is somewhat different from the classical: fuchsian subgroups of $PSL(2, \mathbb{C})$ are usually required to preserves a round disk in $\mathbb{S}^2$.

5.1. **Quasifuchsian groups of codimension 1.**

**Theorem 5.6.** *(A. Marden [70], B. Maskit [74]) Let $G \subset \mathrm{Mob}(\mathbb{S}^2)$ be such that $\Omega(G)$ consists of precisely two components. Then $G$ is 1-quasifuchsian, is topologically conjugate to a 1-fuchsian group and $\bar{M}^3(G) = (\mathbb{H}^3 \cup \Omega(G))/G$ is homeomorphic to an interval bundle over a surface which is 2-fold covered by $\Omega(G)/G$.*

Our goal is to compare the higher-dimensional situation with this theorem. Suppose that $G$ is a quasifuchsian group of codimension 1 in $\mathrm{Mob}(\mathbb{S}^n)$. Then it follows from Jordan separation, that $\Omega(G)$ consists of two components, $\Omega_1, \Omega_2$. After replacing $G$ by an index 2 subgroup we can assume that each $\Omega_i$ is $G$-invariant, hence $M^n(G) = M_1 \cup M_2$, where $M_i := \Omega_i/G$. By the duality, $H_*(\Omega_j) \cong H_*(point)$. Therefore, if $\Omega_i$ is simply-connected, then $\Omega_i$ is contractible.

**Theorem 5.7.** *Suppose that both $M_j$ are compact and both $\Omega_j$ are simply-connected. Then $\bar{M}^{n+1}(G)$ is homeomorphic to $M_1 \times [0,1]$ provided that $n \geq 5$. The same conclusion holds for $n = 4$ provided that $M_1$ is homeomorphic to $M_2$.*

*Proof.* Note that for homological reasons, $W = \bar{M}^{n+1}(G)$ is compact, hence $G$ is convex-cocompact and $W$ defines an h-cobordism between the aspherical manifolds $M_1$ and $M_2$. The fundamental groups of $M_i$ are isomorphic to $G$. Hence, if $n \geq 5$, according to [31], this h-cobordism is (topologically) trivial and thus $W$ is homeomorphic to $M_1 \times [0,1]$. If $n = 4$ one applies the relative version of [31]: we have a homotopy-equivalence

$$(W, \partial W) \to (M_1 \times I, M_1 \cup M_2)$$

which is a homeomorphism on the boundary. Since $W$ is hyperbolic, this homotopy-equivalence can be homotoped to a homeomorphism and the homotopy is constant near the boundary [31]. □

Note that if $n = 3$, then $M_1$ and $M_2$ are diffeomorphic, provided that they admit a hyperbolic structure (which conjecturally always holds). Thus, modulo Thurston's hyperbolization conjecture, each 2-quasifuchsian subgroup of $\mathrm{Mob}(\mathbb{S}^3)$ is topologically conjugate to a 2-fuchsian group. If $n = 4$ the existence of a homeomorphism $M_1 \to M_2$ is an open problem.

**Theorem 5.8.** *For each $n \geq 4$ there are codimension 1 quasifuchsian subgroups $G$ in $\mathrm{Mob}(\mathbb{S}^n)$ which are not isomorphic to fuchsian groups.*

*Sketch of the proof:* Our construction is a slight modification of the examples due to M. Gromov and W. Thurston [41]. Let $n \geq 4$. Gromov and Thurston construct examples of negatively curved compact conformally-flat $n$-manifolds $M^n$ which admit no hyperbolic structure. One can apply their construction to produce examples of *uniformizable* flat conformal manifolds with the same properties. Moreover, $M^n = \Omega_1/G$ and $G$ is convex-cocompact, and $\Omega(G) = \Omega_1 \cup \Omega_2$ is the union of two simply-connected components. Then $\Lambda(G)$ are homeomorphic to $\mathbb{S}^{n-1}$, since the limit set of $G$ is homeomorphic to the ideal boundary of the universal cover of $M^n$. Here is an outline of the construction. Let $\varphi = x_1^2 + ... + x_n^2 - s x_{n+1}^2$ be a quadratic form of the signature $(n,1)$, where $s$ is an algebraic integer such that:

(1) The group $\Gamma$ of automorphisms of $\varphi$ over $\mathbb{Q}(s)$ is a cocompact lattice, $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n) \subset SO(n,1)$.

(2) $\Gamma$ contains a finite cyclic group $\mathbb{Z}_k = \langle\theta\rangle$ which fixes the subspace $\{x : x_1 = x_2 = 0\} \subset \mathbb{R}^{n,1}$; the even number $k = k(s)$ can be made arbitrarily large (by choosing $s$ appropriately).

The group $\Gamma$ is contained in the arithmetic group $\Gamma^0 \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ which is the group of automorphisms of the form

$$\varphi_0 = x_0^2 + x_1^2 + ... + x_n^2 - s x_{n+1}^2$$

over $\mathbb{Q}(s)$. Notice that the group $\Gamma^0$ contains the rotation $\tau$ of order 4 which fixes $\{x : x_0 = x_1 = 0\}$. Let $L_1$ be the hyperplane in the hyperbolic $n$-space, corresponding to $\{x : x_0 = x_1 = 0\}$. The stabilizer of this hyperplane in $\Gamma$ is a subgroup $H_1$ which acts as a lattice there. Now the existence of the rotation $\tau$ implies that the *normal injectivity radius* of $L_1/H_1$ in $\mathbb{H}^n/\Gamma_k$ is at least $\mu_{n+1}$ where $\mu_{n+1}$ is the *Margulis constant* for $\mathbb{H}^{n+1}$.

We now repeat the construction from [41] for sufficiently large $k$.

Let $\Delta = \mathrm{Ker}(\Gamma \to \mathbb{Z}_k)$ The orbifold $\mathbb{H}^n/\Delta$ splits into $k$ pieces if we remove from it the orbit of $L_1/F_1$ under the action of $\mathbb{Z}_k$. The closure of each piece is an orbifold $U_j$ whose boundary is totally geodesic outside of the closed $(n-2)$-dimensional totally geodesic suborbifold $N = (L^0 = \{x : x_1 = x_2 = 0\} \cap \mathbb{H}^n)/J$, where $J$ is the stabilizer of $L^0$ in $\Gamma$ (which acts as an arithmetic lattice on $L^0$). Denote by $E_j$ the subgroup of $\Delta$ corresponding to $U_j$; $F_{j-1,j}, F_{j,j+1}$ are the subgroups corresponding to the totally geodesic components of $\partial U_j - N$. Then remove one piece $U_k$ from $\mathbb{H}^n/\Delta_k$ and glue the rest to form an orbifold $O_k$ identifying the boundary components of $\mathbb{H}^n/\Delta_k - U_k$ via $\theta$. This orbifold admits a metric of negative curvature [41]; $O_k$ admits an action of $\mathbb{Z}_{k-1} = \langle\psi\rangle$ which fixes $N$. Let $\widehat{O}$ denote the quotient orbifold: $\widehat{O} := O_k/\mathbb{Z}_{k-1}$.

Our goal is to construct (for large $k$) a discrete and faithful representation $\rho : \pi_1(\widehat{O}) \to \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$, whose image $G$ is a group with the properties as in our theorem. We choose a unique (up to conjugacy) representation $\rho$ such that:

The restrictions of $\rho$ to the subgroups $E_j$ are induced by conjugation via elements $\alpha_j$ in $\mathbf{M}\mathrm{ob}(\mathbb{S}^n)$.

Then each group $\rho(E_j)$ has an invariant totally geodesic hyperplane $V_j$ in $\mathbb{H}^{n+1}$ so that the angles between $V_j$ and $V_{j+1}$ ($j \in \mathbb{Z}_{k-1}$) tend to $\pi$ as $k \to \infty$. This fact and the fact that the injectivity radii of boundary components of $\partial U_j - N$ are bounded away from zero imply that we can apply Maskit Combination Theorem to the representation $\rho$:

$\rho(\pi_1(\widehat{O}))$ is the HNN extension of $\rho(E_2)$ via a Moebius transformation $\psi$ of finite order ($\psi$ conjugates $\rho(F_{1,2})$ and $\rho(F_{2,3})$). $\square$

The above examples have another interesting property. Let $\Omega^{n+1}$ denote the domain of discontinuity of the group $G$ (regarded as a subgroup of $\mathbf{M}\mathrm{ob}(\mathbb{S}^{n+1})$). Note that $\Omega^{n+1}$ is connected, its fundamental group is infinite cyclic and one can show that $\Omega^{n+1}$ is aspherical. Thus we can apply the topological rigidity results of Farrell and Jones to the quotient manifold $M^{n+1}(G) = \Omega^{n+1}/G$ as follows. The manifold $X_0 = X := M^{n+1}(G)$ is homeomorphic to $\mathbb{S}^1 \times M^n$ (these manifolds are homotopy-equivalent and the manifold $\mathbb{S}^1 \times M^n$ is nonpositively curved, so we can use [32]). The

$(n+1)$-manifold $X$ has a flat conformal structure $K_0$ uniformized by the group $G$. Take the $j$-fold covering $X_j \to X$ which is induced by the $j$-fold covering over $\Omega^{n+1}$ and let $K_j$ denote the pull back of the flat conformal structure $K_0$ to this covering.[3] We thus get an infinite family of new flat conformal manifolds $(X_j, K_j)$ $(j = 0, 1, 2, ...)$. Topological rigidity of Farrell and Jones implies that all $X_j$'s are homeomorphic; results of [64, Essay IV] imply that there are infinitely many diffeomorphic manifolds among $X_j$'s.

**Question 5.9.** Suppose that $M$ is a compact hyperbolic manifold. Is there a finite cover $f : M' \to M$ such that the pull-back map $f^* : H^3(M, \mathbb{Z}/2) \to H^3(M', \mathbb{Z}/2)$ is trivial? (Recall [64] that the group $H^3(M, \mathbb{Z}/2)$ classifies the PL structures on $M$.)

The proof of the following claim is similar to [52].

**Claim 5.10.** *For different $i, j$ the manifolds $(X, K_i)$ and $(X, K_j)$ lie in different connected components of the moduli space $\mathfrak{M}(X)$ of flat conformal structures on $X$. Thus $\mathfrak{M}(X)$ consists of infinitely many connected components.*

We note that K. Scannell in [88] constructs examples of closed hyperbolic 3-manifolds $X$ for which $\mathfrak{M}(X)$ consists of infinitely many connected components.

We now return to our discussion of Kleinian groups, restricting to $n = 3$. Suppose that $G$ is a 2-quasifuchsian group without parabolic elements such that both components of $\Omega(G)$ are simply-connected. Since $G$ is convex-cocompact, it is also Gromov-hyperbolic; hence, according to a theorem of Bestvina and Mess [12], for each component $\Omega_j$ of $\Omega(G)$ there is a homeomorphism of $\overline{\Omega_j} = \Omega_j \cup \Lambda(G)$ to the closed 3-ball. It now follows that the limit set of such group is *tame*, i.e. there is a homeomorphism of $\mathbb{S}^3$ which maps $\Lambda(G)$ to the round sphere.

**Example 5.11.** (B. Apanasov, A. Tetenov [5].) There is a convex-cocompact 2-quasifuchsian group $G \subset \mathrm{Mob}(\mathbb{S}^3)$ whose limit set is a *wild* 2-sphere, i.e. there is no a homeomorphism of $\mathbb{S}^3$ which maps $\Lambda(G)$ to the round sphere. Moreover, one component of $\Omega(G)$ is simply-connected.

### 5.2. 1-quasifuchsian subgroups of $\mathrm{Mob}(\mathbb{S}^3)$.

Given a Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^3)$ whose limit set is a topological circle $C$ we would like to analyze the embedding $C \hookrightarrow \mathbb{S}^3$. It is clear that $C$ could be an unknot in $\mathbb{S}^3$ (i.e. there exists a homeomorphism of $\mathbb{S}^3$ which maps $C$ onto a round circle), take for instance any 1-fuchsian group.

Recall that a topological circle $C$ in $\mathbb{S}^3$ is called *tame* if it is isotopic to a polygonal knot in $\mathbb{S}^3$; if $C$ is not tame, it is called *wild*.

**Proposition 5.12.** *1. If $G$ is a 1-quasifuchsian sugroup of $\mathrm{Mob}(\mathbb{S}^3)$ then either $\Lambda(G)$ is an unknot or it is a wild knot $K$ such that $\pi_1(\mathbb{S}^3 \setminus K)$ is infinitely generated.*
*2. Each 1-quasifuchsian group is geometrically finite.*

[3] Alternatively, one can describe the structures $K_j$ via *grafting* of $(X, K_0)$ along the hypersurface $M^n$.

*Proof.* Since $G$ is 1-quasifuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$, this group is nonelementary. The fundamental group of $M^3 := \Omega(G)/G$ is finitely generated (since $G$ is) and we have the exact sequence:

$$1 \to \pi_1(\Omega(G)) \to \pi_1(M) \to G \to 1.$$

Suppose that $\pi_1(\Omega(G))$ is finitely generated. Then, according to Jaco-Hempel's Theorem [46], $\pi_1(\Omega(G)) \cong \mathbb{Z}$. This immediately excludes tame nontrivial knots (cf. [67]). It remains to exclude wild knots with cyclic $\pi_1(\Omega(G)) \cong \mathbb{Z}$. Note that $M$ is a Seifert manifold (since its fundamental group contains infinite cyclic normal subgroup and $M$ is irreducible). Hence $M$ admits an $\mathbb{S}^1$-action. Lift this action to $\Omega(G)$ and then extend to the entire 3-sphere (so that the fixed point set is the limit set). Raymond's classification [87] of topological $\mathbb{S}^1$ actions on $\mathbb{S}^3$ implies that this $\mathbb{S}^1$-action is topologically conjugate to the orthogonal action, hence $\Lambda(G)$ is a tame unknot. This proves (1). To prove (2) note that the group $G$ acts as a *convergence group* on $\mathbb{S}^1 = \Lambda(G)$; hence this action is topologically conjugate to a fuchsian action $F \curvearrowright \mathbb{S}^1$ on the unit circle (because our groups are torsion-free this follows from [98]), which is always geometrically finite. Since geometric finiteness is an invariant of topological dynamics on the limit set, the group $G$ is geometrically finite. □

**Example 5.13.** (M. Gromov, B. Lawson, W. Thurston, N. Kuiper [39], [66], and M. Kapovich [51, 54].) There are 1-quasifuchsian groups $G \subset \mathbf{Mob}(\mathbb{S}^3)$ such that $\Lambda(G)$ are unknotted but $G$ are not topologically conjugate to 1-fuchsian groups.

The reason is that if $G$ is topologically conjugate to a 1-fuchsian group then $M(G) = \Omega(G)/G$ is a Seifert manifold with the zero Euler number. However there are examples when $M(G)$ is a nontrivial circle bundle over an orientable surface.

**Example 5.14.** (B.Apanasov [4], B.Maskit [74].) There are 1-quasifuchsian groups $G \subset \mathbf{Mob}(\mathbb{S}^3)$ such that $\Lambda(G)$ are wild knots and the fundamental groups of $\Omega(G)$ are infinitely generated.

The idea of the proof: start with a knotted necklace of tangent round balls $B_1, ..., B_m$ in $\mathbb{S}^3$. Let $G'$ be the group generated by the reflections in $\partial B_j$. This group is orientation reversing, but its index 2 subgroup $G$ has the desired properties. To see that $\pi_1(\Omega(G))$ is infinitely generated apply Seifert–Van-Kampen Theorem to the quotient manifold $M^3(G) = \Omega(G)/G$ and conclude that the kernel of $\pi_1(M) \to G$ is not isomorphic to $\mathbb{Z}$.

Very little is known about quasifuchsian groups in $\mathbf{Mob}(\mathbb{S}^n)$ whose limit sets have dimension between 2 and $n-2$. Perhaps the most interesting result here is obtained by I. Belegradek [8] who used the construction from [39] to get

**Example 5.15.** There exist convex-cocompact 2-quasifuchsian subgroups $G_1, G_2 \subset \mathbf{Mob}(\mathbb{S}^4)$ so that:

1) $\Lambda(G_1)$ is a wild 2-sphere in $\mathbb{S}^4$.

2) $\Lambda(G_2)$ is tame but the group $G_2$ is not topologically conjugate to a 2-fuchsian group: $M^4(G_2)$ is a nontrivial circle bundle over a hyperbolic 3-manifold.

## 6. Ahlfors finiteness theorem in higher dimensions: quest for the holy grail

6.1. **The holy grail.** One of the most fundamental results of the theory of Kleinian subgroups of $\mathrm{Mob}(\mathbb{S}^2)$ is the *Ahlfors' Finiteness Theorem* (the "Holy Grail"), which we state here together with its companions:

**Theorem 6.1.** *Suppose that $G \subset \mathrm{Mob}(\mathbb{S}^2)$ is a Kleinian group*[4] *which may have torsion. Then the following hold:*

*1. (L. Ahlfors [2], L. Greenberg [37]) The quotient $O := \Omega(G)/G$ is a complex orbifold of finite conformal type*[5]*. In particular, $\pi_1(O)$ is finitely generated.*

*2. (D. Sullivan [93]) $G$ has only finitely many cusps.*

*3. (M. Feighn and G. Mess [33]) $G$ has only finitely many $G$-conjugacy classes of finite order elements.*

*4. (P. Scott [89, 90]) $G$ is finitely presentable and the orbifold $M = \mathbb{H}^3/G$ is finitely covered by a manifold $\mathbb{H}^3/G'$, which is homotopy-equivalent to a compact 3-manifold.*

*5. (L. Ahlfors) The action of $G$ on $\Lambda(G)$ is* recurrent *with respect to the Lebesgue measure $\mu$ (i.e. for every measurable subset $A \subset \Lambda(G)$ either $\mu(A) = 0$ or $\mu(A \cap g(A)) > 0$ for some $g \in G - \{1\}$).*

**Corollary 6.2.** *If $G$ is as above then;*

*a. For each component $\Omega_0$ of $\Omega(G)$, the limit set of the stabilizer of $\Omega_0$ in $G$ equals $\partial\Omega_0$ (follows directly from (1)).*

*b. Kleinian subgroups $G$ of $\mathrm{Mob}(\mathbb{S}^2)$ are* coherent*, i.e., each finitely generated subgroup of $G$ is also finitely presented (follows from (4)).*

*c. (W. Thurston, see [79]) If $G \subset \mathrm{Mob}(\mathbb{S}^2)$ is geometrically finite with $\Omega(G) \neq \emptyset$ then each finitely generated subgroup $H \subset G$ is geometrically finite as well.*

**Conjecture 6.3.** *If $G$ is as above, then the action of $G$ on $\Lambda(G)$ is ergodic with respect to the Lebesgue measure: each measurable $G$-invariant function on $\Lambda(G)$ is constant a.e..*

This conjecture is known provided that the 3-manifold $\mathbb{H}^3/G$ is *topologically tame*, i.e. is homeomorphic to the interior of a manifold with boundary.

Note, that the conglomerate of assertions presented above contains statements of different nature: algebraic, topological, dynamical. For a while it was hoped that a theorem analogous to Theorem 6.1 can be proven for Kleinian groups in higher dimensions; an attempt to develop analytical technique to achieve this was made by Ahlfors in [1] (see also [82]).

Nearly all algebraic and topological assertions of Theorem 6.1 and the Corollary 6.2 have been disproved in the case of Kleinian groups acting in higher dimensions (M. Kapovich and L. Potyagailo, [63], [62], [56], [84], [85]):

[4]Recall that all Kleinian groups are assumed to be finitely generated.

[5]I.e., as a Riemann surface it biholomorphic to a compact Riemann surface with a finite subset removed; as an orbifold it has only finitely many singular cone-points.

**Theorem 6.4.** *There exist discrete groups $F_1, F_2, F_3, F_4, F_5$ in* $\mathrm{Mob}(\mathbb{S}^3)$ *so that:*

*1. The group $F_1$ is not finitely presentable.*

*2. For each $i$, the manifold $M(F_i) = \Omega(F_i)/F_i$ contains a component with infinitely generated fundamental group.*

*3. $F_2$ is free and has infinitely many cusps (of rank 1).*

*4. $F_3$ is not torsion-free and has infinitely many conjugacy classes of finite order elements.*

*5. $F_4$ is a normal subgroup of a convex-cocompact group $G_4 \subset \mathrm{Mob}(\mathbb{S}^3)$ and satisfies (1), (2) and (4).*

*6. (B. Bowditch, G. Mess* [20]*) The group $F_5$ satisfies (1) and (2) and is contained in a cocompact lattice $G_5 \subset \mathrm{Mob}(\mathbb{S}^3)$.*

*7. Groups $F_i$, $i = 1, ..., 4$ are normal subgroups of geometrically finite groups $G_i$ so that $G_i/F_i \cong \mathbb{Z}$.*

*Remark* 6.5. By modifying $F_3$ one can also construct an example $F_6 \subset \mathrm{Mob}(\mathbb{S}^3)$ such that $\Omega(F_6)/F_6$ contains infinitely many connected components.

All the examples $F_i$ in the above theorem are based upon existence of hyperbolic 3-manifolds $M^3$ of finite volume which fiber over the circle: the groups $F_i$ are obtained by manipulating with the normal surface subgroups in $\pi_1(M^3)$.

**Problem 6.6.** *Find examples similar to $F_i$'s without using hyperbolic 3-manifolds fibered over the circle.*

Note however that although algebra and topology fail, the assertions of dynamical nature (Theorem 6.1, part 5 and Corollary 6.2, part (b)) remain open in higher dimensions. Moreover, an attempt to construct a higher-dimensional counter example to Theorem 6.1 (part 5) along the lines of the examples $F_i$, is doomed to failure:

**Theorem 6.7.** *(K. Matsuzaki,* [77]*) Let $G$ be a geometrically finite subgroup*[6] *in* $\mathrm{Mob}(\mathbb{S}^n)$. *Suppose that $N \subset G$ is a normal subgroup. Then the action of $N$ on its limit set is recurrent.*

Ergodicity fails however for discrete subgroups of $PU(2,1)$ (it probably also fails for Kleinian groups in higher dimensions but an example would be difficult to construct):

**Theorem 6.8.** *There exists a finitely generated (but not finitely presentable!) discrete group $N$ of isometries of complex-hyperbolic 2-plane $\mathbb{CH}^2$ so that the limit set of $N$ is the 3-sphere and the action of $N$ on $\mathbb{S}^3$ is not ergodic.*

*Proof.* There are examples (the first was constructed by R. Livne in his thesis [69], see also [29]) of cocompact torsion-free discrete subgroups $G \subset PU(2,1)$ such that the complex 2-manifold $M = \mathbb{CH}^2/G$ admits a nonconstant holomorphic map $f : M \to S$ to a Riemann surface $S$ of genus $\geq 2$. The fundamental group of the generic fiber of $f$ maps onto a normal subgroup $N$ in $G$, so that $N$ is finitely generated but is not finitely presentable [57]. By lifting $f$ to the universal covers we get a nonconstant holomorphic function

$$\tilde{f} : \mathbb{CH}^2 \to \mathbb{H}^2$$

[6] Actually, the proof also works for subgroups of any rank 1 Lie group.

which is $N$-invariant. Let $h = Re(\tilde{f})$, then $h$ is $N$-invariant, bounded nonconstant harmonic function. Thus $h$ admits an extension of $\mathbb{S}^3 = \partial_\infty \mathbb{CH}^2$, which is $N$-invariant, measurable and a.e. nonconstant. □

6.2. **Groups with small limit sets.** So far, our quest for the holy grail mostly resembles Monty Python's: we are not sure what to look for in higher dimensions (except for the elusive dynamical assertions which will probably fail as well). Nevertheless, there is a glimmer of hope.

We recall that the Hausdorff dimension $\dim_H$ of a bounded subset $E \subset \mathbb{R}^n$ is defined as follows. For each $\alpha > 0$ consider the $\alpha$-Hausdorff measure of $E$:

$$mes_\alpha(E) = \lim_{\rho\to 0} \inf\{\sum_i r_i^\alpha, r_i \le \rho, E \text{ is contained in the union of } r_i\text{-balls}\}.$$

The Hausdorff dimension of $E$ is

$$\dim_H(E) = \inf_\alpha\{\alpha : mes_\alpha(E) = 0\}.$$

Recall that dim stands for the topological dimension. Then, according to [47], for every bounded subset $E \subset \mathbb{R}^n$ one has the inequality

$$\dim(E) \le \dim_H(E).$$

In particular, if $G$ is a finitely-generated Kleinian group with $\dim_H(\Lambda(G)) < 1$ then $G$ is geometrically finite and is isomorphic to a Schottky-type group, see Theorem 3.7. Recall also that the exponent of convergence of a Kleinian group $G \subset \mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ is

$$\delta(G) := \inf\{s > 0 : \sum_{g\in G} e^{-sd(x,gx)} < \infty\},$$

where $d$ is the hyperbolic metric in $\mathbb{H}^{n+1}$. Then, according to [81], [14], one has

$$\delta(G) = \dim_H(\Lambda_c(G)) \le \dim_H(\Lambda(G)).$$

**Conjecture 6.9.** *Suppose that $G$ is a geometrically finite subgroup of* $\mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ *so that $\Lambda(G)$ has topological dimension $< 2$. Then each finitely generated subgroup of $G$ is geometrically finite.*

Note that Conjecture 6.9 holds if $\Lambda(G)$ is 0-dimensional. If Conjecture 6.9 fails for groups with $\dim(\Lambda(G)) = 1$, one can try:

**Conjecture 6.10.** *Suppose that $G$ is a (finitely generated) subgroup of* $\mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ *so that $\Lambda(G)$ has Hausdorff dimension $< 2$. Then $G$ is geometrically finite.*

For $n = 2$, this conjecture is a theorem of Bishop and Jones [14]. A partial generalization of [14] was proven in [24]:

**Theorem 6.11.** *Suppose that $G$ is a (finitely generated)* conformally finite[7] *subgroup of* $\mathbf{M}\mathrm{ob}(\mathbb{S}^n)$ *such that* $\dim_H(\Lambda(G)) < n$. *Then $G$ is geometrically finite.*

[7] I.e., $M^n(G) = \Omega(G)/G$ is compact modulo cusps.

**Theorem 6.12.** *(Y. Shalom [91]) Suppose that $G$ is a geometrically finite subgroup of* $\mathrm{Mob}(\mathbb{S}^n)$ *such that* $\dim_H(\Lambda(G)) < 2$ *and* $N \subset G$ *is a finitely generated normal subgroup. Then $N$ has finite index in $G$. In particular, $N$ is geometrically finite as well.*

On the other hand, the assumption that $\delta(G)$ is small should impose strong restrictions on the algebraic properties of the group $G$.

**Conjecture 6.13.** *Suppose that $G$ is a (finitely generated) Kleinian group in* $\mathrm{Mob}(\mathbb{S}^n)$ *which does not contain parabolic elements. Then* $cd(G) - 1 \leq \delta(G)$.

Note that the weaker assertion: $cd(G) - 1 \leq \dim_H(\Lambda(G))$ is true if $\dim_H(\Lambda(G)) < 1$.

**Theorem 6.14.** *Conjecture 6.13 holds for geometrically finite groups. Indeed, under this assumption, the group $G$ is actually convex-cocompact. Hence, according to* [12], $cd(G) - 1 = \dim(\Lambda(G))$ *and we get:*

$$cd(G) - 1 = \dim(\Lambda(G)) \leq \dim_H(\Lambda(G)) = \dim_H(\Lambda_c(G)) = \delta(G).$$

In the case of equality one actually gets a rigidity theorem:

**Theorem 6.15.** *(Chenbo Yue,* [101], *cf.* [16].*) Suppose that $G$ is convex-cocompact and* $i = \delta(G) = cd(G) - 1$. *Then $G$ is $i$-fuchsian.*

Chenbo Yue states his theorem in the context of $i$-quasifuchsian groups, but his proof actually does not need this assumption.

**Conjecture 6.16.** *Suppose that $G$ is a Kleinian group in* $\mathrm{Mob}(\mathbb{S}^n)$ *whose limit set is not totally disconnected and has Hausdorff dimension* 1. *Then $G$ is* 1*-fuchsian.*

## 7. Deformations of Kleinian groups

Let $\Gamma$ be a (finitely-generated) group, consider the following algebraic variety (usually called, the *representation variety* of $\Gamma$):

$$R(\Gamma, n) := Hom(G, SO(n+1,1)).$$

The group $SO(n+1,1)$ acts on $R(G,n)$ via conjugation and one can form a quotient variety

$$X(\Gamma, n) := R(\Gamma, n)//SO(n+1,1),$$

called the *character variety.* Roughly speaking, the elements of $X(\Gamma, n)$ are represented by conjugacy classes of representations $\rho : \Gamma \to SO(n+1,1)$. This is literally true for "most" representations (the ones for which $\rho(\Gamma)$ does not contain a normal parabolic subgroup[8]), see [49].

We will be mostly interested in representations $\rho \in R(\Gamma, n)$ which have discrete, nonelementary image, however much of our discussion is more general. We start by considering the *local* structure of $X(\Gamma, n)$. Given a Kleinian subgroup $\Gamma \subset \mathrm{Mob}(\mathbb{S}^n)$, we will think of elements $[\rho]$ in a sufficiently small neighborhood of $[id_\Gamma] \in X(\Gamma, n)$ as *small deformations* of $\Gamma$. Given a representation $\rho \in R(\Gamma, n)$ we get the action of $\Gamma$ on the vector space $V_\rho$, which is given by the *adjoint* action $Ad(\rho(\gamma))$, $\gamma \in \Gamma$, of

[8] Such $\rho$'s are called *reductive.*

$\rho(\Gamma)$ on the Lie algebra $so(n+1,1)$ of the group $SO(n+1,1)$. The associated 1-st cohomology group

$$H^1(\Gamma, V_\rho) = Z^1(\Gamma, V_\rho)/B^1(\Gamma, V_\rho)$$

can be regarded as the space of *infinitesimal deformations* of the representation $\rho$. More precisely, if $\rho$ is reductive and $\rho(\Gamma)$ has trivial centralizer in $\mathbf{M}\text{ob}(\mathbb{S}^n)$ then $H^1(\Gamma, V_\rho)$ is isomorphic to the Zariski tangent space at $[\rho]$ of the algebraic variety $X(\Gamma, n)$. If $H^1(\Gamma, V_\rho) = 0$ then $\rho$ is an isolated point of $X(\Gamma, n)$, in which case $\rho$ is called *locally rigid.*

7.1. **Small deformations of 1-quasifuchsian groups.** We recall that the Lie group $\mathbf{M}\text{ob}(\mathbb{S}^n)$ has dimension $d = (n+2)(n+1)/2$. Suppose that $\Gamma$ is 1-quasifuchsian and has no finite normal subgroups (we allow $\Gamma$ to have torsion), then $\Gamma$ is geometrically finite and is isomorphic to a lattice in $\text{Isom}(\mathbb{H}^2)$. After passing to a subgroup of index 2 in $\Gamma$ we can assume that $\Gamma$ embeds discretely in $PSL(2,\mathbb{R})$. If $\Gamma$ contains no parabolic elements then $\Gamma$ has the presentation:

$$\langle a_1, b_1, ...., a_q, b_q, c_1, ..., c_k | [a_1, b_1] \cdot ...[a_q, b_q] \cdot c_1 \cdot ... \cdot c_k = 1, c_j^{r_j} = 1, j = 1, ..., k \rangle.$$

For a representation $\rho : \Gamma \to \mathbf{M}\text{ob}(\mathbb{S}^n)$ we let

$$e_j := d - \dim\{\xi \in so(n+1,1) : Ad \circ \rho(c_j)(\xi) = \xi\};$$

in other words, $e_j$ is the codimension of the centralizer of $\rho(c_j)$ in $\mathbf{M}\text{ob}(\mathbb{S}^n)$. Let $s$ denote the dimension of centralizer of $\rho(\Gamma)$ in $\mathbf{M}\text{ob}(\mathbb{S}^n)$.

**Theorem 7.1.** *(A. Weil* [100]*)*

$$h = \dim\ H^1(\Gamma, V_\rho) = (2q-2)d + 2s + e_1 + ... + e_k. \tag{7.2}$$

*Moreover, if $s = 0$ then $X(\Gamma, n)$ near $[\rho]$ is a smooth $h$-dimensional manifold.*

To better understand the difficulties which one encounters in the case of $i$-fuchsian groups for $i \geq 2$, we consider the *hyperbolic triangle groups* $\Gamma$. These are 1-fuchsian groups with $q = 0$, $k = 3$; every such $\Gamma$ has the presentation

$$\langle c_1, c_2, c_3 | c_1 \cdot c_2 \cdot c_3 = 1, c_j^{r_j} = 1, j = 1, 2, 3 \rangle,$$

where $r_1^{-1} + r_2^{-1} + r_3^{-1} < 1$. Such group embeds discretely into $PSL(2,\mathbb{R})$ and we will denote the image of this embedding by $\Delta = \Delta(r_1, r_2, r_3)$

As a subgroup of $\mathbf{M}\text{ob}(\mathbb{S}^2)$, the group $\Delta$ is locally rigid (which follows from vanishing of $H^1$). Moreover, triangle groups are "superrigid" in $\mathbf{M}\text{ob}(\mathbb{S}^2)$, i.e. every discrete representation of $\Gamma$ into $\mathbf{M}\text{ob}(\mathbb{S}^2)$ is induced by conjugation.

The situation changes somewhat if we consider representations into $\mathbf{M}\text{ob}(\mathbb{S}^3)$. First, let $\rho_0 : \Delta \to \Gamma' \subset \mathbf{M}\text{ob}(\mathbb{S}^3)$ be the embedding obtained as the composition

$$\Delta \subset \mathbf{M}\text{ob}(\mathbb{S}^1) \to \mathbf{M}\text{ob}(\mathbb{S}^2) \to \mathbf{M}\text{ob}(\mathbb{S}^3).$$

For $\rho_0$ we get: $d = 10, s = 1, e_j = 6$ $(j = 1, 2, 3)$. Thus (7.2) tells us that $\Gamma'$ is still locally rigid in $\mathbf{M}\text{ob}(\mathbb{S}^3)$. However, instead of $\rho_0$ we can take a *twisted* extension. Suppose that we can find numbers $m_j \in \mathbb{Z}, 1 < |m_j| < r_j - 1$ $(j = 1, 2, 3)$ such that:

$$m_1^{-1} + m_2^{-1} + m_3^{-1} = 0, \text{ and } \forall j,\ m_j \text{ divides } r_j.$$

(This is satisfied for instance by $m_1 = m_2 = 4, m_3 = -2$ and $r_j = 8$ for all $j$.)

Define a homomorphism $\theta : \Delta \to \mathbb{S}^1$ by sending $c_i$ to the rotation by $2\pi/m_i$. Then define $\rho : \Delta \to SO(2) \times \mathbf{M}\text{ob}(\mathbb{S}^1) \subset \mathbf{M}\text{ob}(\mathbb{S}^3)$ by twisting $\rho_0$ via $\theta$. It is clear that $\rho(\Delta)$ is a 1-fuchsian subgroup in $\mathbf{M}\text{ob}(\mathbb{S}^3)$. If $r_j > 3$ for each $j$, then $e_j = 8$, $s = 1$ and the formula (7.2) gives the dimension $h = 6$ for $H^1(\Delta, V_\rho)$. I do not know if among these *infinitesimal* deformations there are integrable (i.e., tangent to curves in $X(\Delta, 3)$). To decide this one has to analyze the quadratic form

$$\phi : H^1(\Delta, V_\rho) \to H^2(\Delta, V_\rho) = \mathbb{R},$$

given by the cup-product $[\xi] \mapsto [\xi] \cup [\xi]$. Here $\phi([\xi])$ is represented by the 2-cocycle

$$\tau(x, y) = [\xi(x), Ad \circ \rho(x)\xi(y)]$$

where $[,]$ is the Lie bracket. According to [36], $\{\phi = 0\}$ is analytically isomorphic to a neighborhood of $[\rho]$ in $X(\Delta, 3)$; hence it suffices to find a nontrivial 1-cocycle $\xi$ for which $\phi([\xi]) = 0$ to get nontrivial deformations of the representation $\rho$.

On the other hand, every representation $\rho$ of the group $\Delta = \Delta(2, 3, r_3)$ into $\mathbf{M}\text{ob}(\mathbb{S}^3)$ has zero cohomology $H^1(\Delta, V_\rho)$. Thus the group $\Delta$ has only finitely many conjugacy classes of representations into $\mathbf{M}\text{ob}(\mathbb{S}^3)$.

7.2. ***i*-quasifuchsian groups for $i \geq 2$.** In the case of $i$-quasifuchsian groups $\Gamma$ ($i \geq 2$) the existence of nontrivial deformations of representations $\Gamma \to \mathbf{M}\text{ob}(\mathbb{S}^n)$ is not at all clear. Suppose that $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ is a cocompact lattice. Then one has the following sources of deformations of the identity representation $\rho = id : \Gamma \hookrightarrow \text{Isom}(\mathbb{H}^{n+1})$):

(a) *Bending* along closed totally geodesic hypersurfaces in $M^n = \mathbb{H}^n/\Gamma$ (see [49], [65]).

(b) Geometric construction of deformations $\rho$, provided that $\Gamma \subset \text{Isom}(\mathbb{H}^3)$ is a reflection group [55]. In this case one can show that $X(\Gamma, 3)$ is smooth and has dimension $f - 4$, where $f$ is the number of faces of the fundamental polyhedron of $\Gamma$. Similarly, one gets $\dim H^1(\Gamma, V_\rho) = f - n - 3$ for $n \geq 4$, but it is unclear which of these infinitesimal deformations are integrable.

**Example 7.3.** For each $n \geq 4$ there are examples of cocompact torsion-free lattices $\Gamma \subset \mathbf{M}\text{ob}(\mathbb{S}^n)$ for which there exist nonintegrable elements in $H^1(\Gamma, V_\rho)$, see [49].

(c) *Generalized bending* associated with a collection of compact totally-geodesic submanifolds with boundary in $M^n$, see [6][9], [78].

**Example 7.4.** Suppose that $M^n$ is a closed hyperbolic manifold which contains a collection of compact totally-geodesic submanifolds (with connected boundary) $M_i^{n-1}$, $i = 1, ..., p$, which share common boundary $M^{n-2}$. Assume that $M \setminus \cup_i M_i^{n-1}$ consists of $p$ connected components. Then

$$\dim H^1(\pi_1(M^n), V_\rho) \geq p - 3.$$

Moreover (cf. [61]), if all the angles $\alpha_i$ between the adjacent $M_i^{n-1}$, $M_{i+1}^{n-1}$ are $< \pi$ then there is an analytic embedding of the cone

$$\{(x_1, ..., x_{p-3}) : x_1^2 + ... + x_{p-2}^2 - x_{p-3}^2 = 0\}$$

[9]Some of the theorems stated in this paper are probably incorrect since they do not take into account the restrictions on the angles between the totally-geodesic submanifolds.

into $X(\Gamma, n)$ so that the origin maps into $[\rho]$. If one of the angles is $> \pi$ then nothing could be said about existence of integrable infinitesimal deformations. To see role of the inequalities $\alpha_i < \pi$, the reader should first try to do the following. Take a great circle $C$ in the sphere $\mathbb{S}^2$, subdivide $C$ into segments, one of which has length $> \pi$. Then try to "bend" the resulting circular polygon in $\mathbb{S}^2$ preserving the lengths of the edges. This polygonal bending models the generalized bending of hyperbolic manifolds.

**Example 7.5.** Suppose that we have a totally-geodesic immersion $f : M^{n-1} \to M^n$, where $M^{n-1}$ is a closed hyperbolic $(n-1)$-manifold, so that $\nu(x) = |f^{-1}f(x)| \le 2$ for each $x \in M^{n-1}$ and the submanifold $\{x \in M^{n-1} : \nu(x) = 2\}$ consists of two components each of which separates $M^{n-1}$. Then $H^1(\pi_1(M^n), V_\rho) \ne 0$.

And that's all if $n > 3$. Assume now that $n = 3$.

(d) There are examples of "stumping deformations" [3], generalized in [94].

(e) The first obstruction to integrability of infinitesimal deformations is always zero, see [60].

(f) There are examples of $\Gamma$ for which $H^1(\Gamma, V_\rho) = 0$, see [55] (with non-Haken examples) and also [88]. Note that Scannell's [88] examples are Haken and contain incompressible surfaces which are not fibers in a fibration over $\mathbb{S}^1$.

7.3. **Space of discrete and faithful representation.** Let $\mathcal{D}(\Gamma, n) \subset X(\Gamma, n)$ denote the subset corresponding to discrete, faithful and nonelementary representations of $\Gamma$. This subset is known to be closed, see for instance [72].

**Theorem 7.6.** *(J. Morgan, [80].) Suppose that $\Gamma$ does not split as an amalgam over a virtually abelian group. Then $\mathcal{D}(\Gamma, n)$ is compact.*

See also [58] for the proof using the Rips theory. Unfortunately, none of the known proofs of this theorem gives explicit bounds on the "size" of the compact set $\mathcal{D}(\Gamma, n)$.

**Problem 7.7.** *Find a "constructive" proof of Theorem 7.6 for finitely-presented groups. More precisely, given $\Gamma$ with a finite presentation $\langle g_1, .., g_k | R_1, .., R_m \rangle$ find a constant $C$ which depends on $n$, $k, m$ and the lengths of the words $R_i$ so that the following holds. For each $[\rho] \in \mathcal{D}(\Gamma, n)$ there exist a point $x \in \mathbb{H}^{n+1}$ so that*

$$d(x, \rho(g_i)(x)) \le C, i = 1, ..., k.$$

Theorem 7.6 suggests that one should also look for *geometric bounds* on the images of $\rho$ for $[\rho] \in \mathcal{D}(\Gamma, n)$. For instance, suppose that $\Gamma$ is the fundamental group of a closed aspherical manifold $M^{p+1}$. One can look for the *volume bounds* for the representations $\rho$. Namely, given a representation $\rho$ one can try to estimate infimum and supremum of the volume (which we will denote $Vol(f)$) of $M$ in the metrics induced by the maps $f : M^{p+1} \to \mathbb{H}^{n+1}/\rho(\Gamma)$. The upper bounds for the volume are easy because the volumes of geodesic $k$-simplices in $\mathbb{H}^{n+1}$ are bounded from above by $c_k < \infty$. If $[M]$ is the fundamental cycle of $M^{p+1}$ which is represented by $m$ simplices of dimension $p$, then

$$Vol(f) \le m \cdot C_n.$$

The lower bounds are more difficult. We let $Vol(G)$ denote $\inf_f Vol(f)$, where $G := \rho(\Gamma)$. Let $||M||$ denote Gromov norm of $M$. Under our assumptions, $||M||$ is positive (but hard to compute).

**Theorem 7.8.** *(Follows directly from* [38, Theorem 5.38]*). There exists a universal constant $c(p,n)$ such that $Vol(G) \geq c(p,n)||M||$.*

One gets a better estimate in the case of groups isomorphic to lattices. The following is a corollary[10] of the the work of Besson, Courtois and Gallot [10]:

**Theorem 7.9.** *Suppose that $\Gamma$ is a lattice in $\mathrm{Mob}(\mathbb{S}^p)$, $p \geq 2$, $[\rho] \in \mathcal{D}(\Gamma, \mathrm{Mob}(\mathbb{S}^n))$ and let $G := \rho(\Gamma)$. Then*

$$\left(\frac{p}{n}\right)^{p+1} Vol(\mathbb{H}^{p+1}/\Gamma) \leq Vol(G).$$

*Remark* 7.10. I think that the estimate

$$\frac{p}{n} Vol(\mathbb{H}^{p+1}/\Gamma) \leq Vol(G) \tag{7.11}$$

holds for $p = 1$ provided that $\rho$ sends parabolic elements to parabolic elements. It follows from the minimal surface theory that (7.11) holds for $n = 2$.

7.4. **"Pinching vs. Pinching".** Let $\Gamma \subset \mathrm{Mob}(\mathbb{S}^n)$. An element $\gamma \in \Gamma$ is *pinched* by $\rho \in \mathcal{D}(\Gamma, n)$ (such $\gamma$ is also called *an accidental parabolic element for $\rho$*) if $\gamma$ is hyperbolic, but $\rho(\gamma)$ is parabolic.

Suppose now that $n = 2$, $\Gamma \subset \mathrm{Mob}(\mathbb{S}^1)$ is a (torsion-free) 1-fuchsian group. Then $\gamma \in \Gamma \setminus \{1\}$ can be pinched by some $[\rho] \in \mathcal{D}(\Gamma, 2)$ iff $\gamma$ is represented by a simple closed geodesic on $S = \mathbb{H}^2/\Gamma$. More generally, if we are given $C := \{\gamma_1, ..., \gamma_k\} \subset \Gamma$, (where $\gamma_i$'s are pairwise nonconjugate) then $\gamma_i$'s can be pinched simultaneously by some $\rho$ iff the set $C$ can be partitioned as $C = C_1 \sqcup C_2$ so that each $C_i$ can be represented by a system of simple pairwise disjoint closed geodesics. Therefore, $k \leq 2(3q - 3 + p)$, where $S$ has genus $q$ and $p$ punctures.

Now consider the space $\mathcal{D}(\Gamma, 3)$. Then for the example $F_3$ in Theorem 6.4 one can find $\Gamma \subset \mathrm{Mob}(\mathbb{S}^1)$ and an isomorphism $\rho : \Gamma \to F_3$ which sends parabolic elements to parabolic elements. Thus we can pinch simultaneously an infinite set of elements represented by simple closed loops in $S$.

It is also easy to construct examples of pinching of elements of $\Gamma$ represented by *nonsimple* closed geodesics on $S$.

**Definition 7.12.** Let $M^n$ be a complete Riemannian manifold with negative sectional curvature so that for each $x \in M^n$ the sectional curvature at $x$ is between $-a^2$ and $-1$. The optimal constant $-a^2$ is called *the pinching constant* for $M^n$.

Given any finite set $C$ of closed geodesics in $S$ as above one can construct a negatively pinched 4-manifold $M^4$ with the fundamental group $\Gamma$ such that no element of $L$ can be represented by closed geodesic in $M^4$. However, the pinching constants for $M^4$ will depend on $L$ and it is impossible to find a universal pinching constant $a$ (independent on $C$) for which such pinching is possible: this follows from a version

[10] I am grateful to J. Soto for this remark.

of Theorem 7.6 for negatively curved manifolds. In particular, given $n \in \mathbb{N}$, there are finite subsets $C \subset \Gamma$ which cannot be pinched by any $[\rho] \in \mathcal{D}(\Gamma, n)$.

**Conjecture 7.13.** *For every finite set $C = \{g_1, ..., g_k\} \subset \Gamma$, there exists a number $n = n(\Gamma, L)$ such that: the system $C$ can be pinched by some $[\rho] \in \mathcal{D}(\Gamma, n)$.*

7.5. **Topological and algebraic constrains on Kleinian groups.** Sadly, there are only few known algebraic and topological restrictions on Kleinian subgroups in $\mathrm{Mob}(\mathbb{S}^n)$ that do not follow from general restrictions on geometry of complete negatively curved Riemannian manifolds. Residual finiteness and Selberg lemma is one. Other restrictions follow from the compactness theorem 7.6 (see the previous section and also [9]). Below are few more restrictions.

Theorem 7.6 implies:

**Proposition 7.14.** *Consider a pair of compact 3-manifolds $N_1, N_2$, whose interior admits a complete hyperbolic metric of finite volume, so that the boundary of each $N_j$ is a single torus $T_j$. Given a homeomorphism $f : T_1 \to T_2$ let $M_f := N_1 \cup_f N_2$. Then for each $n \in \mathbb{N}$, the number of $M = M_f$'s such that $\pi_1(M)$ admits a discrete embedding in $\mathrm{Mob}(\mathbb{S}^n)$ is finite.*

We recall that a group $\Gamma$ has property (T) if each isometric action of $\Gamma$ on a Hilbert space fixes a point. See [99] for a construction of a large class of examples of Gromov-hyperbolic groups which have property (T).

**Proposition 7.15.** *If $\Gamma$ is an infinite Kleinian*[11] *group in $\mathrm{Mob}(\mathbb{S}^n)$, then $\Gamma$ does not have property (T).*

*Proof.* Suppose that $\Gamma \subset G = \mathrm{Mob}(\mathbb{S}^n)$. It is known that the group $G$ itself does not have property (T), let $G \curvearrowright \mathcal{H}$ be an action on a Hilbert space without a fixed point. If $\Gamma \subset G$ fixes a point $v \in \mathcal{H}$ then C. Moore's theorem on vanishing of matrix coefficients (see for instance [102, Theorem 2.2.20]) implies that $G$ fixes $v$ as well. □

**Corollary 7.16.** *If $L$ is a lattice acting on the quaternionic hyperbolic space $\mathbb{H}\mathbb{H}^k$, $k \geq 2$, then the image of each homomorphism $\rho : L \to \mathrm{Mob}(\mathbb{S}^n)$ fixes a point in $\mathbb{H}^{n+1}$.*

*Proof.* Since $L$ has property (T), the group $\rho(L)$ also has property (T), see [28]. □

Note that this corollary also follows from Corlette's superrigidity theorem [27].

**Theorem 7.17.** *Suppose that $\Gamma \subset \mathrm{Mob}(\mathbb{S}^3)$ is a torsion-free Kleinian group and let $\sigma_j : \Sigma_j \to \mathbb{H}^4/\Gamma$ $(j = 1, 2)$ be $\pi_1$-injective maps of closed oriented surfaces $\Sigma_j$. Then*

$$|\langle \sigma_1, \sigma_2 \rangle| \leq C(\chi(\Sigma_1), \chi(\Sigma_2)),$$

*where $\langle ., . \rangle$ is the algebraic intersection number and $C(x_1, x_2)$ is a universal function (independent of $\Gamma$), see [53].*

[11] Actually, discreteness of $\Gamma$ is irrelevant, it is important for the proof however to assume that $\Gamma$ is not relatively compact in $\mathrm{Mob}(\mathbb{S}^n)$.

**Corollary 7.18.** *Suppose that* $\Gamma_1, \Gamma_2 \subset \mathrm{Mob}(\mathbb{S}^3)$ *are 1-quasifuchsian subgroups which contain no parabolic elements and the subgroup of* $\mathrm{Mob}(\mathbb{S}^3)$ *generated by the* $\Gamma_i$*'s is discrete. Then the algebraic linking number of the limit sets* $\Lambda(\Gamma_1), \Lambda(\Gamma_2)$ *is bounded from above by a number which depends only on the rank*[12] *of* $\Gamma_1$ *and* $\Gamma_2$.

## 8. Generalizations of Kleinian groups

There are several ways to generalize Kleinian groups:

(1) Consider *uniformly quasiconformal groups* of homeomorphisms of $\mathbb{S}^n$, i.e., the groups $G$ for which there exists a constant $K$ such each $g \in G$ is $K$-quasiconformal.

(2) Consider Riemannian $(n+1)$-manifolds $M$ of negatively pinched curvature. The fundamental group $G = \pi_1(M)$ of such manifold acts isometrically on the universal cover $\tilde{M}$. The manifold $\tilde{M}$ admits a natural compactification $\tilde{M} \cup \mathbb{S}^n$ and the action $G \curvearrowright \tilde{M}$ extends to a discrete action $G \curvearrowright \mathbb{S}^n$ on the topological sphere $\mathbb{S}^n$.

By considering (1) and (2) one gets groups of homeomorphisms of $\mathbb{S}^n$ which satisfy the *convergence property*:

Each sequence $(g_i)$ of elements of $G$ contains a subsequence $(g_{i_j})$ so that either $(g_{i_j})$ converges to a homeomorphism of $\mathbb{S}^n$ or there is a pair of points $x, y \in \mathbb{S}^n$ such that $(g_{i_j})$ converges to the constant map $\mathbb{S}^n \setminus \{y\} \mapsto x$ uniformly on compacts in $\mathbb{S}^n \setminus \{y\}$.

Such groups are called *convergence groups*, they were first introduced in [35]. In what follows we will assume that all group actions on spheres are orientation-preserving.

### 8.1. **Uniformly quasiconformal groups.**

When $n = 2$ there is essentially no difference between conformal and uniformly quasiconformal groups (because the Beltrami equation is solvable):

**Theorem 8.1.** *(D. Sullivan, see also* [95]*.) If $G$ is a uniformly quasiconformal group acting on* $\mathbb{S}^2$ *then $G$ is quasiconformally conjugate to a group of Moebius transformations.*

The situation changes dramatically in the dimensions $\geq 3$. The first (nondiscrete) examples of uniformly quasiconformal groups acting on $\mathbb{S}^n$ ($n \geq 3$) for which Theorem 8.1 fails were constructed by P. Tukia [96]. Examples of discrete uniformly quasiconformal groups which are not topologically conjugate to Kleinian groups were produced in [34], [71], [50]. All these examples require nontrivial elements of finite order. For example, take the group $G$ from the Example 5.13, let $\mathbb{S}^1$ be the smooth group of rotations along the fibers of the Seifert manifold $M^3(G)$, pick a subgroup $\mathbb{Z}_2 \subset \mathbb{S}^1$, lift it to $\mathbb{S}^3$ and consider the extension $\Gamma$ of $G$ by $\mathbb{Z}_2$. The group $\Gamma$ is uniformly quasiconformal but cannot be topologically conjugate to a conformal group, since $G$ is not topologically conjugate to a 1-fuchsian group. N. Isachenko [48] constructed an example of a discrete uniformly quasiconformal group $\Gamma$ acting on $\mathbb{S}^3$ which is not isomorphic to any subgroup (including nondiscrete ones) of $\mathrm{Mob}(\mathbb{S}^3)$. A torsion-free discrete uniformly quasiconformal group $\Gamma$ which is not topologically conjugate to a subgroup of $\mathrm{Mob}(\mathbb{S}^3)$ can be constructed by taking a finite extension of a group $G$

[12] I.e. the least number of generators.

in the example 5.13, so that $\Omega(G)/\Gamma$ is a circle bundle over a surface with very large Euler number. The group $\Gamma$ is not topologically conjugate to a subgroup of $\mathbf{M}\mathrm{ob}(\mathbb{S}^3)$ because of Theorem 7.17.

**Question 8.2.** Suppose that $G \curvearrowright \mathbb{S}^n$ is a discrete uniformly quasiconformal group, assume that this action admits uniformly quasiconformal extensions $G \curvearrowright \mathbb{S}^{n+i}$ for each $i \geq 1$ (the bound on the coefficients of quasiconformality may depend on $i$). Is it true that there is an $i \in \mathbb{N}$ and an extension $G \curvearrowright \mathbb{S}^{n+i}$ which is topologically conjugate to a Moebius action?

I suspect that the answer is negative and one can get a counterexample by producing $G \curvearrowright \mathbb{S}^n$ so that $G$ is not isomorphic to a Kleinian subgroup of $\mathbf{M}\mathrm{ob}(\mathbb{S}^k)$ for any $k$. Groups with the property (T) may do the job.

8.2. **Negatively curved manifolds and convergence groups.** Consider a complete Riemannian manifold $M = M^{n+1}$ with negatively pinched sectional curvature. Then the ideal boundary of its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{S}^n$ and the action of $G = \pi_1(M)$ on $\mathbb{S}^n$ is a discrete convergence action. Let $\Omega(G)$ and $\Lambda(G)$ again denote the domain of discontinuity and the limit set for the action of $G \curvearrowright \mathbb{S}^n$. Consider the convex hull $N := Hull(\Lambda(G)) \subset \tilde{M}$ of $\Lambda(G)$, then the ideal boundary of $N$ equals $\Lambda(G)$, see [19]. One can define geometrically finite and convex-cocompact actions actions $G \curvearrowright \tilde{M}$ in the same way as for the Kleinian groups. Similarly, one can define geometrically finite and convex-cocompact convergence actions if $G \curvearrowright \mathbb{S}^n$ using Theorem 2.2.

The next theorem follows from Thurston's hyperbolization theorem (see [58, 83]):

**Theorem 8.3.** *If $n = 2$ and $G \curvearrowright \mathbb{S}^2$ is geometrically finite then this action is topologically conjugate to a Kleinian group action.*

In contrast, the following problem is open:

**Question 8.4.** Suppose that $G \curvearrowright \mathbb{S}^2$ is a convex-cocompact convergence group action. Is it true that $G$ is topologically conjugate to a Kleinian group?

One can show (cf. [59]) that the above question is equivalent to

**Conjecture 8.5.** *(J. Cannon) Suppose that $G \curvearrowright \mathbb{S}^2$ is a convex-cocompact convergence group action so that $\Lambda(G) = \mathbb{S}^2$. Then $G$ is isomorphic to a Kleinian subgroup of* $\mathbf{M}\mathrm{ob}(\mathbb{S}^2)$.

The geometrically infinite actions $G \curvearrowright \tilde{M}^3$ on the universal covers of negatively pinched 3-manifolds are also difficult to handle. It is not even clear if the analogue of Ahlfors' finiteness theorem holds in this context:

**Conjecture 8.6.** *The area of $\partial(Hull(\Lambda(G))_\epsilon)/G$ is finite. Equivalently one can ask if $\partial Hull(\Lambda(G))/G$ is compact modulo cusps.*

**Theorem 8.7.** *Suppose that $G$ is finitely generated, torsion-free and freely indecomposable. Then Conjecture 8.6 holds for $G$.*

*Sketch of the proof:* Let $N := Hull(\Lambda(G))/G$. Suppose that a component $X$ of $\partial N$ contains embedded discs $D_j$ of arbitrary large radius. (This is equivalent to the negation of the statement of Conjecture 8.6.) Then there are open neighborhoods $U_j$ of $D_j$ in $N$ which are still contractible. Now consider $X$ as a part of a *geometrically infinite end* $E$ of the manifold int$(N)$ and find a sequence of closed geodesics $\gamma_k$ in $N(G)$ that "exits" this end. Then one can use the arguments of Bonahon [15] and Canary [23] (which actually work in the category of negatively pinched manifolds) and find a sequence of *ruled simplicial surfaces* $F_k$ which "exit" the end $E$. The surfaces $F_k$ have uniformly bounded area and therefore cannot contain the subsets $F_k \cap U_j$ of arbitrary large area. □

Given this, one can now find a Kleinian group $G' \subset \mathrm{Mob}(\mathbb{S}^2)$ is isomorphic to $G$ so that there exists a homeomorphism $h : \bar{M}^3(G) \to \bar{M}^3(G')$. One can also choose $G'$ so that the ending laminations of the geometrically infinite ends $E_j \subset \tilde{M}/G$ match the ending laminations of the manifold $\mathbb{H}^3/G'$. Now, it is likely that one can use the arguments of Brock, Canary and Minsky from their proof of the ending lamination conjecture to conclude that the above groups $G$ and $G'$ are topologically conjugate.

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Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090
*E-mail address*: kapovich@math.utah.edu