Ricci Flow and the Uniformization of Surfaces

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What is a Surface?

- A surface is a closed (i.e. compact and boundaryless) 2-dimensional manifold.
- Examples: Sphere, Torus, Connected Sum of Tori.

- A surface is determined uniquely by its genus, roughly how many "holes" are in the surface, and the orientation.
- From the genus ,g, we can define an invariant called the Euler characteristic $\chi(M^2) = 2 - 2g.$
Foundations from Geometry

For a differentiable manifold, $M$, and a differentiable function $\alpha : (-\epsilon, \epsilon) \rightarrow M$ such that $\alpha(0) = p \in M$. The tangent vector to the curve $\alpha$ at $t = 0$ is a function $\alpha'(0) : C^\infty(M) \rightarrow \mathbb{R}$ given by:

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt} \big|_{t=0}$$

- The set of all tangent vectors to $M$ at a point $p$ is called the tangent space of $M$ at $p$ denoted as $T_pM$.
- $T_pM$ is a vector space such that $\dim T_pM = \dim M$.
- We choose $\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\}$ as an orthonormal basis for $T_pM$. We will write $\partial_i$ as $\frac{\partial}{\partial x_i}$.

A Riemannian metric on a manifold $M$ is a correspondence which associates to each point $p$ of $M$ an inner product $<,>$ on $T_pM$. The metric is $C^r$ if the function $g(X, Y) : p \mapsto g_p(X_p, Y_p)$ is of class $C^r$.

- Example 1: $\mathbb{R}^n$ is a Riemannian manifold with metric given by $< e_i, e_j >= \delta_{ij}$ where $e_i = (0, \cdots, 1, \cdots, 0)$. 

Ricci Flow
Example 2: $S^2 \subset \mathbb{R}^3$ is a Riemannian manifold with metric given by $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix}$.

This makes $S^2$ a Riemannian manifold with the round metric.

Example 3: Let $\mathbb{H}^n = \{(x_1, \cdots, x_n) : x_n > 0\}$ with the metric $g_{ij} = \frac{\delta_{ij}}{x_n}$, this makes $\mathbb{H}^n$ into a Riemannian manifold known as hyperbolic space.

From the metric we can define the covariant derivative $\nabla \frac{\partial}{\partial t} Y = (\frac{dY^k}{dt} + \Gamma^k_{ij} \dot{x}^i Y^j) X_k$ where $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

For $\mathbb{R}^n$ the metric is constant therefore $\Gamma^k_{ij} \equiv 0$ which implies that the covariant derivative coincides with the usual derivative.
Curvature of a Surface

Define a map $\mathbf{R}(X, Y)Z : X(M) \mapsto X(M)$ by $\mathbf{R}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$. This is known as the curvature of $M$.

- Equivalently, since $[\partial_i, \partial_j] = 0$ we obtain $\mathbf{R}(\partial_i, \partial_j)\partial_k = (\nabla\partial_j \nabla\partial_i - \nabla\partial_i \nabla\partial_j)\partial_k$
- Set $\mathbf{R}(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l$ and $\langle \mathbf{R}(\partial_i, \partial_j)\partial_k, \partial_l \rangle := \mathbf{R}(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}$
- One can show the following identities are true:
  - $R_{ijkl} = R_{ijkl}^m g_{ml}$, $R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$, $R_{ijkl} = R_{klij}$.
  - For a surface $\{1, 2\}$ are the only possibilities for $i, j, k, l$ therefore the only nonzero term of the curvature tensor is $R_{1212}$.

- If $M = \mathbb{R}^n$ we have $\mathbf{R}(X, Y)Z \equiv 0$ therefore we can think of $\mathbf{R}$ as a way of measuring how much $M$ deviates from being Euclidean.
- By the second statement we see that the curvature measures the non-commutativity of the covariant derivative.
Different Kinds of Curvature

Define the sectional curvature $K_\sigma := \langle R(u, v)v, u \rangle$ where $\sigma$ is a two-dimensional subspace of $T_pM$ and $\{u, v\}$ is an orthonormal basis of $\sigma$. The scalar curvature is defined as $R = g^{jk}R^i_{ijk}$.

- On surfaces we have that $K_\sigma = \frac{R}{2}$. The scalar curvature completely characterizes the curvature in dimension 2.
- When the scalar curvature is positive at a point, the volume of a small ball about that point has a smaller volume than a ball of the same radius in Euclidean space. If the scalar curvature is negative point, then the volume of a small ball about that point has a larger volume than a ball of the same radius in Euclidean space. This means that balls in hyperbolic space grow faster than balls in Euclidean space.

Two metrics $g$ and $h$ on a surface $M^2$ are conformally related if there exists a function, $u$, such that $g = e^{2u}h$.

- For example, stereographic projection of a sphere onto the plane with a point at infinity is a conformal map.
- The scalar curvatures of two conformal metrics are related by:
  $$R_g = e^{-2u}(-2\Delta_h u + R_h)$$
Geometry vs. Topology of a Surface

Gauss-Bonnet Theorem: \(4\pi \chi(M^2) = \int_{M^2} Rd\mu\).

- This formula tells us that the total curvature of a surface is invariant under topological changes.
- The following is topologically \(S^2\) but is not round. The curvature has mixed sign but by the Gauss-Bonnett formula we know the total curvature is \(8\pi\).
Ricci Flow on Surfaces

- Define \( r := \left( \int_{M^2} Rd\mu \right) / \left( \int_{M^2} d\mu \right) \) and consider the following PDE, called the normalized Ricci flow, on \((M^2, g_0)\) that varies the metric in time:

\[
\frac{\partial}{\partial t} g = (r - R)g
\]

\[g(0) = g_0\]

- Under the normalized Ricci flow on a surface we have:

\[
\frac{\partial}{\partial t} R = \Delta R + R(R - r)
\]  \(\text{(1)}\)

- This PDE depends on the Euler characteristic of the surface. Our goal is to show how this works with nonpositive Euler characteristic.

- Equation (1) is known as a reaction-diffusion equation.
Reaction-Diffusion Equations

- The Laplacian term denotes diffusion of $R$, if the RHS only contained the Laplacian, the equation would be a heat equation. If the equation only contained this term we would expect $R$ to tend to a constant as $t \to \infty$.

- The quadratic term promotes concentration of $R$. If the equation contained only this term (1) would be an ODE and the solution would blow up in finite time for any initial condition satisfying $R(0) > \max\{r, 0\}$.

- How the scalar curvature evolves depends on which term dominates.
Uniformization Theorem

**Theorem 1.** If \((M^2, g_0)\) is a closed Riemannian surface, there exists a unique solution \(g(t)\) of the normalized Ricci flow

\[
\frac{\partial}{\partial t} g = (r - R)g
\]

(2)

\[g(0) = g_0\]

The solution exists for all time. As \(t \to \infty\) the metrics \(g(t)\) converge uniformly in any \(C^k\)-norm to a smooth \(g_\infty\) of constant curvature.

- This implies that any surface admits a canonical geometry. Moreover, it is a classification of surfaces into three families—constant positive, zero, or negative curvature.
- The Ricci flow still hasn’t provided a complete proof of the Uniformization theorem for positive Euler characteristic. R. Hamilton gives a proof but he uses the fact that \(S^2 \setminus pt.\) is conformal to the plane hence requires Uniformization.
A Priori Bounds of the Curvature on Surfaces

For any solution \((M^2, g(t))\) of (2) on a compact surface, there exists a constant \(C > 0\) depending only on the initial metric such that:

- If \(r < 0\) then:
  \[
  r - Ce^{rt} \leq R \leq r + Ce^{rt}
  \]

- If \(r = 0\) then:
  \[
  -\frac{C}{1 + Ct} \leq R \leq C
  \]
Theorem 2. If \((M^2, g_0)\) is a closed Riemannian surface, a unique solution \(g(t)\) of (2) exists for all time and satisfies \(g(0) = g_0\).

Idea of the Proof: By Shi, bounds on the curvature imply a priori bounds on all its derivatives for a short time. Short time existence results tell us that the lifetime of a maximal solution \((M^2, g(t))\) is bounded below by \(\frac{c}{\max_{M^n} |R[g_0]|_{g_0}}\). Long-time existence results then imply that the flow cannot be extended past \(T < \infty\) only if

\[
\lim_{t \to T} \left( \sup_{x \in M^n} |R(x, t)| \right) = \infty
\]

The bounds on \(R\) imply that a unique solution exists for all time.

- (Uniform Equivalence) The solution, \(g(t)\), to (2) is uniformly equivalent to the initial metric. More precisely, there exists a constant \(C > 1\) depending only on \(g(0)\) such that for as long as a solution exists,

\[
\frac{1}{C} g(0) \leq g(t) \leq C g(0)
\]
Convergence when $\chi(M) < 0$

What do we know so far:

- By theorem 2 a solution $g(t)$ exists for $0 < t < \infty$.
- The metrics $g(t)$ are all uniformly equivalent.
- There exists a constant $C > 0$ depending only on the initial metric, $g_0$, such that $R$ is exponentially approaching its average in the sense that:
  \[ |R - r| < Ce^{rt} \]

What is really going on!!:

- We start with some arbitrary surface with more than one hole (i.e. genus greater than one.) and some arbitrary metric.
- We evolve the metric on the surface according to (2) with initial data $g_0$.
- We have already shown that under this evolution the metric is exponentially approaching a metric with constant curvature $r$.
- In some sense the evolution is making our surface canonical.

All that remains is to show that all derivatives of $R$ are dying exponentially.
Assume that \((M^2, g(t))\) is a solution of the normalized Ricci flow with \(r < 0\) then we obtain the following estimates on the derivatives of the curvature after long and arduous computations.

- \(|\nabla R|^2 \leq C_1 e^{rt/2}\) with \(C_1 > 0\)
- \(|\nabla \nabla R|^2 \leq C_2 e^{rt/2}\) with \(C_2 > 0\)
- \(\sup_{x \in M^2} |\nabla^k R(x, t)|^2 \leq C_k e^{rt/2}\) for each positive integer \(k\) and \(C_k < \infty\) is a constant for each \(k\).

This completes the proof if \(\chi(M) < 0\).
Convergence when $\chi(M) = 0$

The idea is the same as $\chi(M) < 0$ but we can only get uniform convergence in any $C^k$ norm to a smooth constant-curvature metric $g_\infty$ as $t \to \infty$.

- $\sup_{x \in M^2} | R(x, t) |$ tends to zero as $t \to \infty$.
- $\sup_{x \in M^2} | \nabla^k R(x, t) |^2 \leq \frac{C_k}{(1 + t)^{k+2}}$

This implies uniform convergence in any $C^k$ norm to a metric of constant curvature.
One Dimension Higher

Instead of three model geometries there are eight model geometries.

- Three geometries with constant sectional curvature
  - $S^3$
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$

- Two geometries with product metrics.
  - $S^2 \times \mathbb{R}$
  - $\mathbb{H}^2 \times \mathbb{R}$

- Three manifolds $(M, g)$ where $M$ is a simply connected Lie group with a left-invariant metric.
  - The Heisenberg group, strictly upper-triangular $3 \times 3$ matrices
  - $\mathbb{R}^2 \times \mathbb{R}\{0\}$ where the action of $t \in \mathbb{R}\{0\}$ is given by the matrix: 
    \[
    \begin{pmatrix}
    t & 0 \\
    0 & t^{-1}
    \end{pmatrix}
    \].
  - $\widetilde{PSL}(2, \mathbb{R})$. 
Conjecture 3. (Thurston) Let $M$ be a closed, orientable, prime 3-manifold. Then there is an embedding of a disjoint union of 2-tori and Klein bottles $\bigsqcup_i T^2_i \subset M$ such that every component of the complement admits a locally homogeneous Riemannian metric of finite volume.

- The manifold is prime if it can not be presented as a connected sum in a non-trivial way, where the trivial way is $P = P \# S^3$.
- One can formulate the above as a prime 3-manifold $M$ has a decomposition along incompressible tori and Klein bottles into pieces whose interiors admit finite volume locally homogeneous metrics.
- A tori in a 3-manifold is said to be incompressible if its fundamental group injects into the fundamental group of the 3-manifold.
Geometrization Conjecture implies Poincare Conjecture

- Suppose we have a prime homotopy 3-sphere \( \Sigma \) that satisfies the conclusion of the Geometrization Conjecture.
- Since \( \pi_1(\Sigma) = \{1\} \), \( \Sigma \) has no incompressible tori or incompressible Klein bottles, and hence the decomposition of \( \Sigma \) must be trivial.
- Again since \( \pi_1(\Sigma) \) is trivial the homogenous model for \( \Sigma \) must be compact.
- But the only compact homogeneous model space is \( S^3 \), hence \( \Sigma \) is diffeomorphic is \( S^3 \).
Perelman’s Idea

Classify finite time singularities that develop inside the manifold.

- **Type 1**: Components of the manifold whose metrics are shrinking down in a controlled manner.

- **Type 2**: A long thin tube diffeomorphic to $S^2 \times (-\epsilon, \epsilon)$ or a long thin tube with a cap of positive curvature on the end.
Singularities

- Go to the singular time remove all regions of the first type and do the 'surgery’ near the 'large’ ends of the thin long tubes to cap them off with standard metrics on the disk.
- The topological effect of these surgeries is to remove components which are known to satisfy Thurston’s Geometrization Conjecture and to do surgery on the other components implementing a connected sum decomposition.
- Then take the manifold after surgery as the initial condition for continuing the Ricci flow.
- Repeat this process producing a flow with surgery defined for all positive time with only finitely many surgeries in any finite time interval.
- Perelman showed if the manifold has finite fundamental group then the Ricci flow with surgery becomes extinct after finite time and there is no need for any analysis as $t \to \infty$. 

Ricci Flow
The End