Complements and local singularities in birational geometry

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- Without further notice, we work over an algebraically closed field k of characteristic zero, e.g., the field of complex numbers $\mathbb{C}$.

Linear systems

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For which positive integers n can we pick a "distinguished representative" $G \in |nD|$?

Question (Effective Iitaka fibration)

For which positive integers m, the map defined by $|mD|$ is birational to the Iitaka fibration of D? e.g. if D is big, when $|mD|$ defines a birtaional map?

Singularities in birational geometry

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- X is called ϵ-lc if $(X, 0)$ is ϵ-lc, etc.

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Example

Let X be a cone over a rational curve of degree n. Then $\mathrm{tmld}(X) = \frac{2}{n}$, X is $\frac{2}{n}$-lc but not $\frac{2}{n}$-klt.

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- As a starting point, we want to look into the case when $B = 0$.

The case when X is of general type

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Theorem (Hacon-McKernan 06, Takayama 06, Tsuji)

Let X be a smooth projective variety of dimension d such that K_X is big. Then there exists a positive integer $m_1 = m_1(d)$ such that $|m_1 K_X|$ defines a birational map.

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Theorem (Hacon-McKernan-Xu 14)

Let (X, B) be a projective lc pair of dimension d such that $K_X + B$ is big and the coefficients of B belong to a DCC set Γ. Then there exists a positive integer $m_2 = m_2(d, \Gamma)$ such that $|m_2(K_X + B)|$ defines a birational map.

- DCC: every descending chain stabilizes, e.g. $\{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\}$ is a DCC set, but $\{\frac{1}{n} \mid n \in \mathbb{N}^+\}$ is not.
- $|m_2(K_X + B)| := |\lfloor m_2(K_X + B) \rfloor|$.

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Theorem (Birkar-Zhang 16)

Let X be a smooth projective variety such that $\kappa(X) \geq 0$. Let $W \to Z$ be an Iitaka fibration of K_X from a resolution of X and F a very general fiber of $W \to Z$. Then there exists a positive integer m depending only on

- $\dim X$, *the dimension of* X,
- β_F, *the middle Betti number of the canonical cover of* F, *and*
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- The theorem can also be generalized to the class of lc pairs, but for technical complexity reasons, it has never been written down.

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Let F be a klt Calabi-Yau variety (i.e. $K_F \sim_{\mathbb{Q}} 0$) of dimension $\leq d$. Does there exist a positive integer I depending only on d such that $IK_F \sim 0$?

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- We will pick this question up later.

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  3. When X is (weak) Fano, i.e. X is klt and $-K_X$ is ample (big and nef).

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Let $X_{n+1} \subset \mathbb{P}(1,1,1,n)$ be a general hypersurface of degree $n+1$. Then X_{n+1} is klt Fano, but $|-mK_{n+1}|$ does not define a birational map whenever $m < \frac{n}{2}$.

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Theorem (Birkar 19)

Let d be a positive integer and ϵ a positive real number. Then there exists $m = m(d, \epsilon)$, such that for any ϵ-lc Fano type variety X of dimension d, $|-mK_X|$ defines a birational map.

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Theorem (BAB, Birkar 20)

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Theorem (BAB, Birkar 20)

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- We also have the following counter-example for pairs:

Failure for effective birationality

- Although Birkar's result is strong, it is later covered by his proof of the Borisov-Alexeev-Borisov (BAB) conjecture:

Theorem (BAB, Birkar 20)

Let d be a positive integer and ϵ a positive real number. Then any ϵ-lc Fano type variety of dimension d belongs to a bounded family.

- We also have the following counter-example for pairs:

Example (Han-L 20)

There exist $\frac{3}{10}$-lc projective surface pairs (X, B_n) such that $-(K_X + B_n)$ is ample, B_n has DCC coefficients, but $|\lfloor -m(K_X + B_n)\rfloor|$ and $-|\lfloor m(K_X + B_n)\rfloor|$ do not define birational maps for any $m < n$.

Non-vanishing and complements

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For a klt Fano variety X, without the ϵ-lc assumption, can we still find a "distinguished element" $G \in |-mK_X|$ satisfying certain good properties?

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Let d be a positive integer. Then there exists a positive integer $n = n(d)$, such that for any klt variety X of Fano type, there exists $G \in |-nK_X|$ such that $(X, \frac{1}{n}G)$ is lc. In particular, $|-nK_X|$ is non-empty.

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- $(X, \frac{1}{n}G)$ is called an *n-complement* of $(X, 0)$.
- What can we say when we start with a pair (X, B) rather than a variety X?

Complements

Definition (Complements)

Let n be a positive integer and (X, B) a pair. An n-complement of (X, B) is a pair (X, B^+), such that

1. (X, B^+) is lc,
2. $n(K_X + B^+) \sim 0$, and
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- For the condition (2), we require linear equivalence, not $\mathbb{Q}$-linear equivalence or $\mathbb{R}$-linear equivalence.
- The purpose for condition (3) is to guarantee that $|-\lfloor n(K_X + B)\rfloor|$ is non-empty once an n-complement exists.

Shokurov's complement conjecture

Conjecture (Shokurov 00)

Assume that

1. (X, B) *be an lc pair of dimension* d *of Fano type,*
2. *the coefficients of* B *belong to a DCC set* Γ*, and*
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Then there exists an n*-complement* (X, B^+) *of* (X, B) *for some* n *which belongs to a finite set.*

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- In particular, there exists a uniform n, depending only on d and Γ, such that $|-\lfloor n(K_X + B)\rfloor|$ is not empty.
- This theorem can be strengthened to the relative case.

Remarks

Theorem (Han-L-Shokurov 19)

Let $X \to Z$ be a contraction. Assume that

1. (X, B) *be an lc pair of dimension* d,
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3. (X, B) *is of Fano type over* Z, *and*
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Then for any point $z \in Z$, *there exists an* n*-complement of* (X, B) *over a neighborhood of* z *for some uniform* $n = n(d, \Gamma)$.

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- The theorem above was shown when
  - $\dim X \leq 2$ and the coefficients of B belong to the standard set ([Shokurov 00]),
  - $\dim X = 3$ and the coefficients of B belong to a finite rational set ([Prokhorov-Shokurov 09]), and
  - when the coefficients of B belong to a finite rational set ([Birkar 19]).

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  4. Log Calabi-Yau fibrations ([Birkar 18]).
- In the rest of the talk, I will talk about the application of our theorem on complements to the study of local singularities questions. In this case, Birkar's result is not strong enough, while our result remains useful.

Minimal log discrepancies

Minimal log discrepancies

- Given an lc pair (X, B) and a log resolution $f : Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$. Recall that $\mathrm{tmld}(X, B)$ is 1 minus the maximal coefficient of B_Y.

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- Now given an lc germ $(X \ni x, B)$ and a sufficiently high log resolution $f : Y \to X \ni x$ such that $K_Y + B_Y = f^*(K_X + B)$. We let

$$\mathrm{mld}(X \ni x, B) := \min\{1 - \mathrm{mult}_E B_Y \mid f(E) = x\}$$

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Conjecture (ACC conjecture for mlds, Shokurov 88)

Let $(X \ni x, B)$ be an lc germ of fixed dimension such that the coefficients of B belong to a DCC set Γ. Then mld$(X \ni x, B)$ belongs to an ACC set.

- ACC: every increasing chain stabilizes, e.g. $\{\frac{1}{n} \mid n \in \mathbb{N}^+\}$ is an ACC set, but $\{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\}$ is not.

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- Only very few cases for the ACC conjecture is known: we know the surfaces case ([Alexeev 93, Shokurov 94]), the toric case ([Borisov 97, Ambro 06]), and few special cases in: dimension 3, quotient singularities, fixed germs.

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- To prove the ACC conjecture for mlds in full generality, we expect to combine our method and the toric method together.

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- The ACC conjecture for mlds was usually considered harder than the ACC for ϵ-lc thresholds, yet the theory of complements tells us that they are likely to be equivalently difficult.

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- (X, B^+) is lc, so (Y, B_Y^+) is lc, hence B_Y^+ has many good properties.

Other applications of the complement theorem

For audience with potential interest, we gather a list of other known applications of our theorem on complements. We remark that these results depend on the complements theorem for arbitrary DCC coefficients (rather than finite rational coefficient).

1. The ACC for mlds for exceptional singularities and singularities admitting an ϵ-plt blow-up ([Han-L-Shokurov 19]).
2. The ACC for complete regularity thresholds ([Han-L-Shokurov 19]).
3. The study on normalized volumes ([Han-Y.Liu-Qi 20]).
4. The study on the effective adjunction conjecture on lc-trivial fibrations ([Li 20]).
5. The study on generalized minimal log discrepancies ([Chen-Gongyo-Nakamura 20](preprint to appear)).
6. Some boundedness results on Fano varieties ([Chen 20]).

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Let F be a klt Calabi-Yau variety (i.e. $K_F \sim_{\mathbb{Q}} 0$) of dimension $\leq d$. Does there exist a positive integer I depending only on d such that $IK_F \sim 0$?

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2. *This result implies the following: there exists a uniform positive integer I, such that for any klt Calabi-Yau threefold F, $IK_F \sim 0$.*

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- "$IK_F \sim 0$" can be rephrased as "$(F, 0)$ is an I-complement of itself". In this case, F is not of Fano type but the boundedness of complements still holds.

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- "$IK_F \sim 0$" can be rephrased as "$(F, 0)$ is an I-complement of itself". In this case, F is not of Fano type but the boundedness of complements still holds.
- We may ask if the boundedness of complements holds for pair admitting lc Calabi-Yau structures (not necessarily of Fano type).

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- Regrettably, the boundedness of complements in dimension ≥ 4 is still widely open.

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Conjecture (Boundedness of complements for non-Fano type varieties)

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- This is the question we mentioned above, which is only known in dimension ≤ 3.
- As a corollary, this conjecture implies the boundedness of indices for K_X, where X is an lc Calabi-Yau variety of fixed dimension.

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Conjecture (Boundedness of ϵ-lc complements)

Let d be a positive integer, ϵ a positive real number, and Γ a DCC set. Then there exist an integer n and a positive real number ϵ' depending only on d, ϵ and Γ satisfying the following. Assume that

1. *(X, B) is an ϵ-lc pair of dimension d,*
2. *the coefficients of B belong to Γ,*
3. *X is of Fano type over Z, and*
4. *$-(K_X + B)$ is nef over Z.*

Then for any point $z \in Z$, there exists an n-complement (X, B^+) of (X, B) such that (X, B^+) is ϵ'-lc.

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- This question is known only for surfaces and curves and when $\dim Z = 0$ (by BAB). When $0 < \dim Z < \dim X$, this question is related to the M^{c}Kernan-Shokurov conjecture. When $\dim Z = \dim X$, this question is related to the ACC for ϵ-lc thresholds.

Thank you!

Key ideas in the proof of the complements theorem

- To prove our theorem on complements, one of the key observation is the existence of various *uniform rational polytopes*.
- We show the existence of uniform $\mathbb{R}$-complementary rational polytopes (in particular, it implies the existence of uniform lc rational polytopes), uniform anti-pseudo-effective rational polytopes, etc.
- Unlike Shokurov's style rational polytopes (e.g. nefness of $K_X + B$ when B varies in $\operatorname{Supp} B$), our uniform rational polytopes only depend on the dimension and the coefficients of B, and do **not** depend on X.
- As the existence of rational polytope style results have many applications, (e.g. Shokurov's polytope in the proof of [Birkar-Cascini-Hacon-M^{c}Kernan 10]), we also expect the uniform rational polytopes to be useful.