Burnside’s Orbit Counting Lemma

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Motivating Example

How many ways is there to fill a tic-tac-toe board with 5 “X”s and 4 “O”s? For example:

\[
\begin{array}{ccc}
X & O & X \\
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X & X & O \\
\end{array}
\]

Answer:
\[
\binom{9}{4} = 9 \cdot 8 \cdot 7 \cdot 6 / 4! = 126
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But we may want to compute the answer up to symmetry, i.e. we wish to consider

\[
\begin{array}{c|c|c}
X & O & X \\
X & O & O \\
X & X & O \\
\end{array}
\]

to be the same as

\[
\begin{array}{c|c|c}
X & O & X \\
O & O & X \\
O & X & X \\ \end{array}
\]
Group actions

Definition

A group of symmetries acting on a set $S$ is a collection $G$ of bijections from $S$ to itself satisfying

1. $id : S \to S$ is in $G$
2. If $g \in G$ then $g^{-1} \in G$.
3. If $g, h \in G$, then $g \circ h \in G$.
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The fixed points of a group element $g \in G$ are

$$\text{Fix}(g) = \{s \in S : g(s) = s\}$$
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Definition
The orbit of an element $s \in S$ is

$$G_s := \{g(s) : g \in G\} \subset S$$
Back to our example

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- A quarter turn.
- A half turn.
- A three quarters turn.
- A horizontal reflection.
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- A vertical reflection.
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- A vertical reflection.
- Two diagonal reflections.
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- A quarter turn.
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One can check that this is a group action.
Example of an orbit

The boards

X | O | X
---+---+---
O | O | X
O | X | X

O | O | X
---+---+---
X | O | O
X | X | X
X | X | O
X | O | O
X | O | O
X | O | X
X | X | X
O | O | X
O | O | X
O | X | X
X | O | X
X | O | O
But not all orbits are the same size:

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\hline
O & X & O \\
\hline
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\end{array}
\]
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\[
\begin{array}{ccc}
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O & X & O \\
X & O & X \\
\end{array}
\]

This orbit has only one element.
The Main Result

Our question can be rephrased as “How many orbits are there?”
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**Theorem (Burnside’s Lemma)**

The number of orbits is equal to the average number of fixed points of elements of $G$, i.e.

$$
\text{# of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|
$$
Let’s count fixed points

- Everything is fixed by the identity: 126.
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- For a quarter rotation, (and a three quarter rotation), we must have

\[
\begin{array}{ccc}
A & B & A \\
B & C & B \\
A & B & A \\
\end{array}
\]

\text{Since there are an odd number of Xs, we must have C=X.}

Then we have two choices: either A = X and B = O or vise versa.
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  \begin{array}{ccc}
  X & O & X \\
  O & X & O \\
  X & O & X \\
  \end{array}
  \quad \quad \quad
  \begin{array}{ccc}
  O & X & O \\
  X & X & X \\
  O & X & O \\
  \end{array}
  \]
AABBCCDDE, so we must pick $E=X$ and two of $A, B, C, \text{or } D$ to be $X$, so $\binom{4}{2} = 6$ choices.
Vertical (or horizontal) reflections

\[
\begin{array}{ccc}
A & D & A \\
B & E & B \\
C & F & C \\
\end{array}
\]

AABBCCCDEF, so we can either:

- Pick X to be one of D,E,F and two of A,B,C: \(3 \times 3 = 9\) choices
- Pick D,E,F to all be X, and then one of A,B,C to be X: 3 choices.

So we have 12 fixed points.
Finally, diagonal reflections

For the diagonal reflections, we have

\begin{array}{c|c|c}
A & B & D \\
\hline
C & E & B \\
F & C & A \\
\end{array}

AABBCDEF, so it is again 12.
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C & E & B \\
F & C & A
\end{array}
\]

AABBCDEF, so it is again 12.
Now we can compute

\[
\frac{1}{8} (126 + 2 \cdot 2 + 6 + 4 \cdot 12) = 23
\]
Problem

Suppose you have two indistinguishable coins with indistinguishable sides. You have \( k \) colors of paint and can paint each side of a coin a single color. How many different things can you do? (Answer will be a function of \( k \).)
Coins

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Suppose you have two indistinguishable coins with indistinguishable sides. You have $k$ colors of paint and can paint each side of a coin a single color. How many different things can you do? (Answer will be a function of $k$.)

$S$ is the set with $k^4$ elements consisting of all pairs of painted coins if you remember left/right/top/bottom.
There are $2^3 = 8$ symmetries— you can swap the coins, and flip each one.
Coins

- id – fixes $k^4$.
- 2 ways to flip one – fixes $k^3$ (flipped coin must be the same color on both sides).
- swap the left and right – fixes $k^2$ (tops and the bottoms must be the same color).
- swap and then flip both – fixes $k^2$ (tops and the bottoms must be the same color).
- flip both – fixes $k^2$ (each coin must be all one color).
- 2 ways to swap and then flip one – fixes $k$ (all sides must be the same color).

Hence the number is $k^4 + 2k^3 + 3k^2 + 2k^8$.

(Exercise: Prove that the polynomial above takes integer values on integers.)
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Hence the number is

$$\frac{k^4 + 2k^3 + 3k^2 + 2k}{8}.$$
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$$8$$

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Some values

\[ N(k) = \frac{k^4 + 2k^3 + 3k^2 + 2k}{8} \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( N(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>55</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
</tr>
<tr>
<td>6</td>
<td>231</td>
</tr>
<tr>
<td>7</td>
<td>406</td>
</tr>
<tr>
<td>8</td>
<td>666</td>
</tr>
<tr>
<td>9</td>
<td>1035</td>
</tr>
<tr>
<td>10</td>
<td>1540</td>
</tr>
<tr>
<td>11</td>
<td>2211</td>
</tr>
<tr>
<td>12</td>
<td>3081</td>
</tr>
<tr>
<td>100</td>
<td>12753775</td>
</tr>
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  \hline
  \( k \) & \( N(k) \) \\
  \hline
  1 & 1 \\
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  \hline
\end{tabular}

At 2 seconds per side, this brings the total to 4.8 years, or just slightly over 100 years.
Problem

You have 5 spoons: three differently colored spoons and two indistinguishable lizard spoons. You have three differently colored bowls. Put one spoon in each bowl, and leave 2 out. The two left-out spoons are unordered. How many ways can you do this?
Solution 1

Let $S$ be the set of 120 possible orderings of five distinct spoons. Let $\ell$ be the function that swaps the two lizard spoons, and $n$ be the function that swaps the two spoons that are not in a bowl. Then our group is $G = \{\text{id}, \ell, n, \ell \circ n\}$.

- $\text{id}$ fixes all 120.
- $\ell$ fixes nothing.
- $n$ fixes nothing.
- $\ell \circ n$ fixes the arrangements with both lizard spoons outside. There are $3! = 6$ ways to arrange the other spoons, and the lizard spoons could be in either order, for a total of $6 \times 2 = 12$.

Hence the number of orbits is $\frac{1}{4}(120 + 12) = 132/4 = 33.$
Solution 2

Let $S$ be the set of possible orderings of 5 spoons with the two lizard spoons indistinguishable. Then we only need the group $G = \{\text{id}, n\}$, where again $n$ switches the two spoons that are not in bowls.

How big is $S$? First, choose where to put the lizard spoons: there are $\binom{5}{2} = 10$ choices. Then put the other three in some order—there are 6 choices. Hence $|S| = 60$.

- $\text{id}$ fixes all 60.
- $n$ fixes the arrangements with both lizard spoons out. There are 6 of these.

Hence the number of orbits is $\frac{1}{2}(60 + 6) = 33$. 
Solution 3 (no Burnside’s)

- Case 1: Lizard spoons both out: 6
- Case 2: One lizard spoon in, one out: 3 choices for the in lizard spoon, 3 choices for the colored spoon to go out, 2 choices to order the colored spoons that go in. So $3 \times 3 \times 2 = 18$.
- Case 3: Two lizard spoons in: 3 choices for a colored spoon to go in a bowl and 3 choices for which bowl to put it in. So $3 \times 3 = 9$.
- Total: 33
A different example

How many possible six sided dice can you make? (consider two dice the same if one can be rotated to match the other)
Solution 1

Start with a blank dice.

- Put 1 on some side. There are no choices here.
- Put some number opposite of 1. There are 5 choices here.
- Put the smallest remaining number somewhere. There are no choices here.
- But now the dice has an orientation. So placing the last 3 numbers has $3 \cdot 2 \cdot 1 = 6$ choices.

So the answer is 30.
Free actions

**Definition**

A group of symmetries $G$ acts freely if no element $g \in G$ has a fixed point (except the identity).

In this case, every orbit has size equal to the size of $G$, and the number of orbits is $|S|/|G|$. 
Solution 2

There are 24 symmetries of the cube: put some face of the cube on top (6 choices), then choose one of 4 rotations of that face).
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These symmetries act freely on the set of numbered dice with orientation. There are $6! = 720$ of these, so the number of orbits is $720/24 = 30$. 

Other examples

- Painting the sides of cubes, tetrahedrons, icosohedrons, etc.

- Counting the number of matrices with entries in a finite set, up to permutations of rows and columns.

- Counting the number of distinct ways to put stickers on a Rubik's cube.

- Counting the number of "distinct" peg jumping game set ups.
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