SOME ADDITIONAL EXPERIMENTS

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The purpose of this note is to report on a very simple experiment. Let C be an elliptic curve over $\mathbb{Q}$. By Mordell's Theorem we know that, as a group, $C \simeq \mathbb{Z}^r \oplus T$, that is it contains infinitely many rational points if r, the rank, is positive.

Now, consider the reduction of the curve C to $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$, and let $N_p = \#C(\mathbb{Z}_p)$. By Hasse's Theorem we know that

$$|p + 1 - N_p| < 2\sqrt{p}$$

so that, roughly speaking, $N_p \sim p + 1$. In order to understand how N_p differs from this "expected value" we will consider the "density" $D_p = \frac{N_p}{p}$.

Morally, at least, a curve with many rational points (that is, large $\#C(\mathbb{Q})$), is expected to have many points over $\mathbb{Z}_p$. We want to show this with a few examples. We will consider curves of the form

$$y^2 = x^3 + ax^2 + bx + c$$

with a, b and c as shown on Table 1

Curve	a	b	c	rank
C_0	0	1	0	0
C_1	0	13	0	1
C_2	0	14	0	2
C_3	337	337^2	337^3	3
C_{14}	0	402599774387690701016910427272483	0	14

TABLE 1. Elliptic curves

The following function, in some sense, measures how much N_p deviates from the "medium value" p.

$$F(x) = \prod_{p \leq x, p\, prime} \frac{N_p}{p}$$

Figure 1 shows the graph of F corresponding to the curves C_0 and C_1.

As you can see, there is a significant difference between the behavior of the two curves. As expected, C_1 seems to have more points than C_0, perhaps reflecting the fact its rank is 1 while that of C_0 is 0. Actually, using Nagel-Lutz's Theorem it is easy to check that $C_0(\mathbb{Q}) \simeq \mathbb{Z}_2$.

Next we see what happens as we consider curves of higher rank. Figure 2 shows the graph of F for C_1, C_2 and C_3.

We see that, again, all functions are, roughly, increasing and that the curves with higher rank have significantly bigger values of F. Notice the scale on the F-axis, compared to Figure 1.

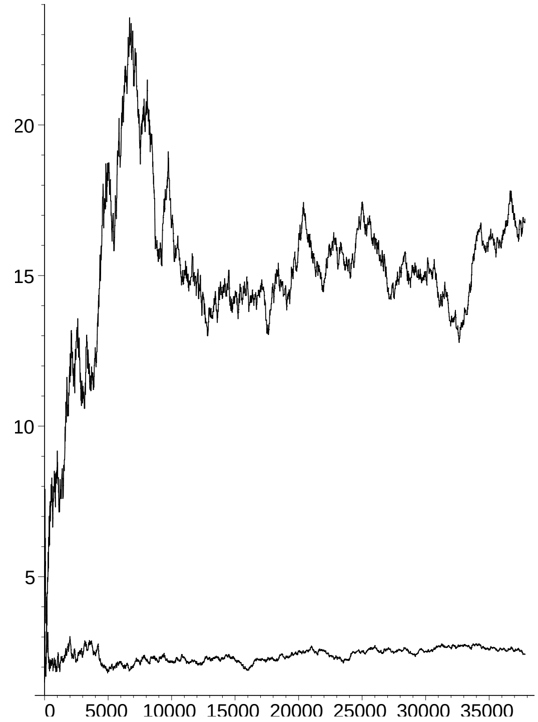

FIGURE 1. Graph of $F(x)$ for C_0 (below) and C_1 (above)

Last, Figure 3 shows the graph of F for C_{14}. Again, notice the scale on the F-axis and thus, how much bigger the values of F are compared with those of the other curves.

To conclude, let me mention that the function $(F(x))^{-1}$ of the elliptic curve C is somehow related to the value of the L-function, $L(C, 1)$. Thus, we see that for a curve of rank 0, the limiting value of $F(x)^{-1}$ is a non-zero number. On the other hand, for all the other curves (with rank > 0) $F(x)^{-1}$ seems to approach to 0. In fact, the bigger the rank, the faster $F(x)^{-1}$ approaches 0. As you have guessed, this is the content of the Birch and Swinnerton-Dyer Conjecture that says that the order of the zero of $L(C, s)$ at $s = 1$ should compute the rank of the curve C.

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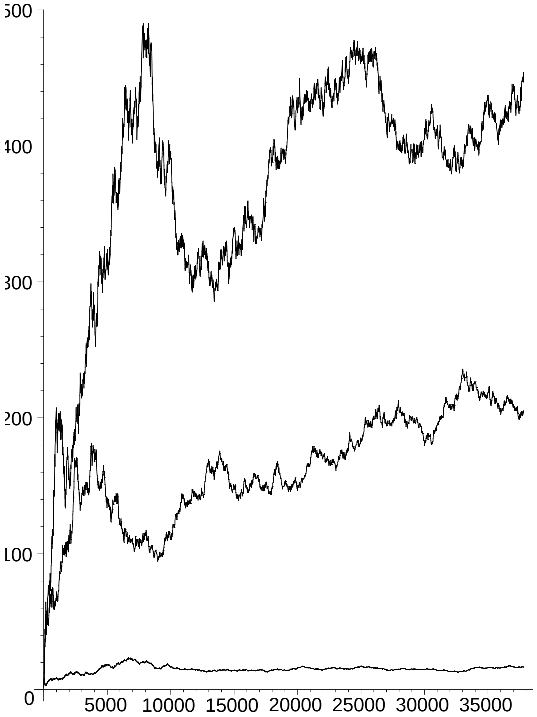

FIGURE 2. Graphs of $F(x)$ for the curves C_1 (bottom), C_2 (middle) and C_3 (top)

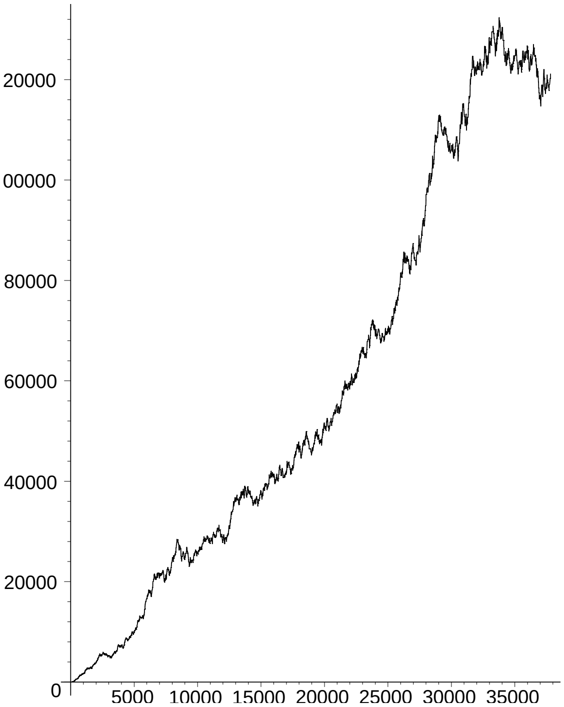

FIGURE 3. Graph of $F(x)$ for the curve C_{14}