

## Problems to think about when preparing for Test 1

MATH 3160-1 - FALL 2002

*Note:* These are some interesting problems. They provide additional practice for Test 1 but, beware: *they do not necessarily cover all the topics to be evaluated by the test.*

1. Prove that  $z$  is real or pure imaginary if and only if  $(\bar{z})^2 = z^2$ .

For  $z = x + iy$  the condition  $(\bar{z})^2 = z^2$  translates into  $(x - iy)^2 = (x + iy)^2$ . Now, a little algebra says that this last condition is equivalent to  $x^2 - y^2 - 2xyi = x^2 - y^2 + 2xyi$  or, simplifying,  $-2xy = 2xy$ . But, this last condition is the same as  $4xy = 0$ , which is the same as saying that either  $x = 0$  or  $y = 0$ . Now, we realize that  $x = 0$  says that  $z$  is a pure imaginary number while  $y = 0$  says that  $z$  is a real number, just as we wanted to show!

2. For  $z_0 = (1 - i)(\sqrt{3} + i)$  and  $z_1 = \frac{1-i}{\sqrt{3}+i}$

- (a) Find the modulus, argument (all possible values), and the principal argument.

We could rewrite  $z_0$  and  $z_1$  as  $a + ib$  and then find all the information. Instead, we will find the modulus and argument of  $1 - i$  and  $\sqrt{3} + i$  and then, using the properties of the modulus and argument, derive those numbers for  $z_0$  and  $z_1$ .

$$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2},$$
$$|\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$$

For the arguments we have:

$$\arg(1 - i) = \arctan\left(\frac{-1}{1}\right) = \arctan(-1) = -\frac{\pi}{4} + 2k\pi$$
$$\arg(\sqrt{3} + i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + 2k\pi$$

At this point it is very important to check that the arguments match the complex numbers: at least we can see in which quadrant each number is and the argument should correspond to that quadrant. For example,  $1 - i$  is in the fourth quadrant and  $-\frac{\pi}{4}$  agrees with that.

Now we use the properties:

$$|z_0| = |(1 - i)(\sqrt{3} + i)| = |1 - i||\sqrt{3} + i| = \sqrt{2} \cdot 2 = 2\sqrt{2},$$
$$|z_1| = \left|\frac{1 - i}{\sqrt{3} + i}\right| = \frac{|1 - i|}{|\sqrt{3} + i|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

For the arguments (since  $\arg(u \cdot v) = \arg(u) + \arg(v)$  and  $\arg(u/v) = \arg(u) - \arg(v)$ ):

$$\arg((1 - i)(\sqrt{3} + i)) = \arg(1 - i) + \arg(\sqrt{3} + i) = -\frac{\pi}{4} + \frac{\pi}{6} + 2k\pi = -\frac{\pi}{12} + 2k\pi$$
$$\arg\left(\frac{1 - i}{\sqrt{3} + i}\right) = \arg(1 - i) - \arg(\sqrt{3} + i) = -\frac{\pi}{4} - \frac{\pi}{6} + 2k\pi = -\frac{5\pi}{12} + 2k\pi$$

These are all the possible arguments. To determine the principal argument we have to pick the one that is in  $(-\pi, \pi]$ . Then  $\text{Arg}(z_0) = -\frac{\pi}{12}$  and  $\text{Arg}(z_1) = -\frac{5\pi}{12}$ .

- (b) Write the exponential (or trigonometric) form.

With all the information we have just computed this is immediate:

$$z_0 = 2\sqrt{2}e^{-\frac{\pi}{12}i},$$

$$z_1 = \frac{1}{\sqrt{2}}e^{-\frac{5\pi}{12}i}.$$

- (c) Sketch a graph showing  $z_0$  and  $z_1$  on the plane.

The graph is shown in Figure 1.

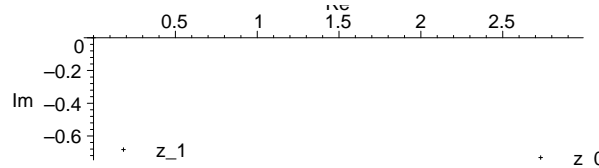


Figure 1: Exercise 2c

3. Find all the fifth roots of  $-1 - i$  and sketch a graph showing their location.

Let  $z$  be a fifth root of  $-1 - i$ . Then,  $z^5 = -1 - i$ , so that

$$|z| = \sqrt[5]{|-1 - i|} = \sqrt[5]{\sqrt{(-1)^2 + (-1)^2}} = \sqrt[10]{2} \simeq 1.0718$$

We have  $\arg(-1 - i) = -\frac{3\pi}{4} + 2k\pi$  (*Note:* we have  $\arctan(\frac{-1}{-1}) = \arctan(1) = \frac{\pi}{4}$ , but our point is in the third quadrant, so a correct value of the argument is  $\frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ ). Then

$$\arg(z) = \frac{-\frac{3\pi}{4} + 2k\pi}{5} = \begin{cases} -\frac{3\pi}{20}, \\ \frac{5\pi}{20}, \\ \frac{13\pi}{20}, \\ \frac{21\pi}{20}, \\ \frac{29\pi}{20}. \end{cases}$$

The graph of the roots is shown in Figure 2.

4. Find the domain of  $f(z) = \frac{z^2 - 2}{2z^3 + i}$ .

Being a rational function, it is well defined everywhere, except at the zeroes of the denominator. So we want to solve

$$2z^3 + i = 0 \Rightarrow z^3 = -\frac{i}{2}.$$

Thus,

$$|z| = \sqrt[3]{\left|-\frac{i}{2}\right|} = \sqrt[3]{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}},$$

$$\arg(z) = \frac{-\frac{\pi}{2} + 2k\pi}{3} = \begin{cases} -\frac{\pi}{6}, \\ \frac{3\pi}{6}, \\ \frac{7\pi}{6}. \end{cases}$$

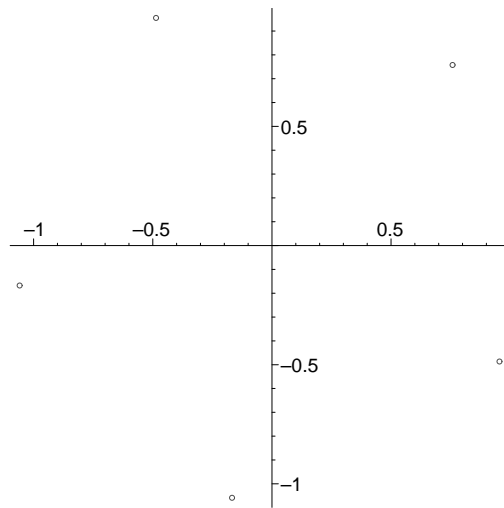


Figure 2: Exercise 3

Therefore, the domain of  $f$  is  $Dom(f) = \mathbb{C} - \left\{ \frac{1}{\sqrt[3]{2}} \exp(-\frac{\pi}{6}i), \frac{1}{\sqrt[3]{2}} \exp(\frac{\pi}{2}i), \frac{1}{\sqrt[3]{2}} \exp(\frac{7\pi}{6}i) \right\}$ .

5. Let

$$U = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 2 \text{ and } |z| \leq 10\}$$

and

$$V = \{z \in \mathbb{C} : 1 < |z| < 3 \text{ and } |\operatorname{Im}(z)| > 2\}$$

(a) Sketch a graph of  $U$  and  $V$ .

Both graphs are shown in Figure 3.

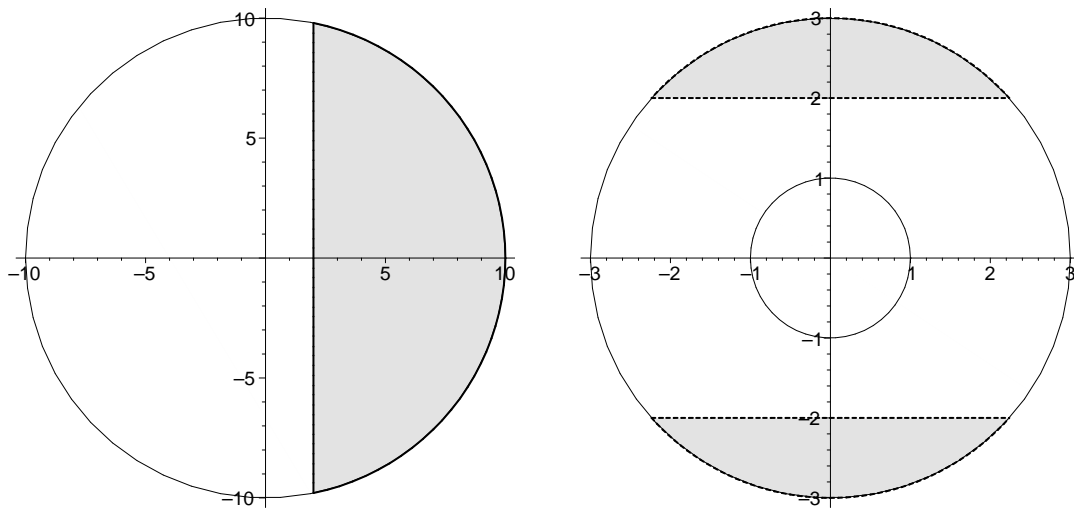


Figure 3: Exercise 5a

(b) Discuss if each set has the following properties:

i. Is open.

$U$  is not open. For instance, the point 2 is not interior in the sense that no disk centered at it is (completely) contained in  $U$ .

$V$  is open since all its points are interior (we see on the graph that no boundary point is in  $V$ ).

ii. Is closed.

$U$  is closed because it contains all its boundary points (contained in the circular arc and the vertical segment).

$V$  is not closed. For instance,  $2i$  is a boundary point of  $V$  (every disk centered at  $2i$  contains both points in  $V$  and not in  $V$ ) and is not in  $V$ .

iii. Is connected.

$U$  is connected: every pair of points can be joined with a polygonal line (actually, a single segment suffices).

$V$  is not connected because it is not possible to join with a (continuous) line any point on the “upper component” with another on the “lower component”.

iv. Is bounded.

Both  $U$  and  $V$  are bounded since they lie inside disks of radii, say 11.

v. Is a domain.

A domain is a set that is open and connected.  $U$  is not open, while  $V$  is not connected. So neither of them is a domain.

6. Compute the following limits, if they exist. Otherwise give an argument of why they don't exist.

(a)  $\lim_{z \rightarrow i} \frac{z^3 - i}{z + i}$

We use the properties of limits, since the denominator doesn't go to zero:

$$\lim_{z \rightarrow i} \frac{z^3 - i}{z + i} = \frac{\lim_{z \rightarrow i} z^3 - i}{\lim_{z \rightarrow i} z + i} = \frac{i^3 - i}{i + i} = \frac{-2i}{2i} = -1$$

(b)  $\lim_{z \rightarrow \infty} \frac{z}{\bar{z}}$

We compute the limit as  $z \rightarrow \infty$  by replacing  $z$  with  $\frac{1}{z}$  and the limit  $z \rightarrow 0$ :

$$\lim_{z \rightarrow \infty} \frac{z}{\bar{z}} = \lim_{z \rightarrow 0} \frac{\frac{1}{z}}{\frac{1}{\bar{z}}} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z}$$

but this limit doesn't exist, as we can see by approaching 0 through the real axis (where the limit is 1) and the imaginary axis (where the limit is  $-1$ ).

(c)  $\lim_{z \rightarrow 2i} \frac{(z-2i)(z+i)}{z^2+4}$ .

If we want to use the properties of limits, we see that the limit of the denominator is 0, and we can't proceed in this way. But, the numerator also tends to 0, which says that we can use some algebraic simplification:

$$\lim_{z \rightarrow 2i} \frac{(z-2i)(z+i)}{z^2+4} = \lim_{z \rightarrow 2i} \frac{(z-2i)(z+i)}{(z-2i)(z+2i)} = \lim_{z \rightarrow 2i} \frac{z+i}{z+2i}$$

But now, since the denominator doesn't go to 0, we can apply the properties and conclude:

$$\lim_{z \rightarrow 2i} \frac{(z-2i)(z+i)}{z^2+4} = \lim_{z \rightarrow 2i} \frac{z+i}{z+2i} = \frac{3i}{4i} = \frac{3}{4}.$$

7. Check the continuity of

$$f(z) = \begin{cases} \frac{z-1}{z^2-1}, & \text{if } z \neq \pm 1 \\ \frac{1}{2}, & \text{if } z = 1 \\ 0, & \text{if } z = -1 \end{cases}$$

at  $z = -1$ ,  $z = 0$  and  $z = 1$ . In each case state clearly what you want to check and then prove it.

$f$  is continuous at  $z_0$  if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ . So, we first need to check if  $\lim_{z \rightarrow -1} f(z) = f(-1)$ . We have:

$$\lim_{z \rightarrow -1} f(z) = \lim_{z \rightarrow -1} \frac{z-1}{z^2-1}$$

and we see that the denominator goes to 0 while the numerator goes to  $-2 \neq 0$ . Thus, the limit is  $\infty$ , different from  $f(-1) = 0$ , and we conclude that  $f$  is not continuous at  $-1$ .

For  $z = 0$  we have to check if  $\lim_{z \rightarrow 0} f(z) = f(0)$ . We have

$$\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \frac{z-1}{z^2-1} = \frac{\lim_{z \rightarrow 0}(z-1)}{\lim_{z \rightarrow 0}(z^2-1)} = \frac{-1}{-1} = 1$$

On the other hand,  $f(0) = 1$ , and we conclude that  $f$  is continuous at 0.

Finally, at  $z = 1$ , we have to check  $\lim_{z \rightarrow 1} f(z) = f(1)$ . We see that the denominator goes to 0, but so does the numerator, and we can simplify:

$$\lim_{z \rightarrow 1} f(z) = \lim_{z \rightarrow 1} \frac{z-1}{z^2-1} = \lim_{z \rightarrow 1} \frac{z-1}{(z-1)(z+1)} = \lim_{z \rightarrow 1} \frac{1}{z+1} = \frac{1}{2}$$

which coincides with  $f(1)$  and we conclude that  $f$  is continuous at 1.

8. Let

$$f(x+iy) = \frac{1}{x^2+y^2}((x^4-y^4+x) + i(2xy(x^2+y^2)-y)).$$

Use the Cauchy-Riemann equations (and additional conditions) to determine where  $f$  defines an analytic function. Find  $f'$  where it is defined.

We see that both  $u$  and  $v$  are “good” (have continuous first derivatives) everywhere except at the origin (this is because we have quadratic terms in the denominator but linear terms in the numerators). So we want to check the Cauchy-Riemann equations in  $\mathbb{C}^* = \mathbb{C} - \{0\}$ .

We patiently compute the required derivatives:

$$\begin{aligned} u_x &= \frac{(4x^3+1)(x^2+y^2) - (x^4-y^4+x)2x}{(x^2+y^2)^2} = \frac{2x^5+4x^3y^2+2xy^4+y^2-x^2}{(x^2+y^2)^2} \\ u_y &= \frac{-4y^3(x^2+y^2) - (x^4-y^4+x)2y}{(x^2+y^2)^2} = \frac{-4y^3x^2-2y^5-2x^4y-2xy}{(x^2+y^2)^2} \\ v_x &= \frac{(6x^2y+2y^3)(x^2+y^2) - (2xy(x^2+y^2)-y)2x}{(x^2+y^2)^2} = \frac{2x^5+4x^3y^2+2xy^4+y^2-x^2}{(x^2+y^2)^2} \\ v_y &= \frac{(2x^3+6xy^2-1)(x^2+y^2) - (2xy(x^2+y^2)-y)}{(x^2+y^2)^2} = \frac{2x^4y+4x^2y^3+2y^5+2xy}{(x^2+y^2)^2} \end{aligned}$$

Finally we check that  $u_x = v_y$  and  $u_y = -v_x$ . The conclusion is that  $f$  is analytic on  $\mathbb{C}^*$ .

9. For  $f(x + iy) = \arctan(\frac{y}{x}) - i \ln(\sqrt{x^2 + y^2})$  let  $u = \operatorname{Re}(f)$ .

(a) Use the Cauchy-Riemann equations to verify that  $f$  is analytic on  $x > 0$ .

As long as we stay away from  $z = 0$  the imaginary part has no problems. On the other hand, the real part of this function has no problems as long as  $x > 0$ . So, the differentiability/continuity conditions are satisfied (in the given region).

We compute the derivatives:

$$(\operatorname{Re}(f))_x = (\arctan(\frac{y}{x}))_x = \frac{1}{1 + (\frac{y}{x})^2} \cdot (-\frac{y}{x^2}) = -\frac{y}{x^2 + y^2}$$

$$(\operatorname{Re}(f))_y = (\arctan(\frac{y}{x}))_y = \frac{1}{1 + (\frac{y}{x})^2} \cdot (\frac{1}{x}) = \frac{x}{x^2 + y^2}$$

$$(\operatorname{Im}(f))_x = (-\ln(\sqrt{x^2 + y^2}))_x = -\frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = -\frac{x}{x^2 + y^2}$$

$$(\operatorname{Im}(f))_y = (-\ln(\sqrt{x^2 + y^2}))_y = -\frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y = -\frac{y}{x^2 + y^2}$$

and check that the Cauchy-Riemann conditions hold.

(b) Check explicitly that  $u$  is a harmonic function (i.e., compute the second derivatives and check the corresponding equation).

Since  $u = \operatorname{Re}(f)$  we already know its first derivatives. We compute the required second derivatives:

$$u_{xx} = (u_x)_x = (-\frac{y}{x^2 + y^2})_x = -\frac{-2xy}{(x^2 + y^2)^2},$$

$$u_{yy} = (u_y)_y = (\frac{x}{x^2 + y^2})_y = \frac{-2xy}{(x^2 + y^2)^2}$$

and we immediately see that  $u_{xx} + u_{yy} = 0$ , so that  $u$  is harmonic for  $x > 0$ .

Remark: In this exercise we computed explicitly everything because it was requested. But remember that we knew from the previous item that  $f$  was analytic, so that without any further computation we could have asserted that  $u$  was harmonic.

(c) Find  $v$ , a harmonic conjugate of  $u$ .

Since  $v$  is the harmonic conjugate of  $u$  we have to impose the Cauchy-Riemann equations. We start by fixing  $x$  and, eventually change variables to  $t = x^2 + y^2$ , so that  $dt = 2ydy$ :

$$\begin{aligned} v &= \int v_y dy = \int u_x dy = \int -\frac{y}{x^2 + y^2} dy = -\int \frac{1}{2t} dt = -\frac{1}{2} \ln(t) + C(x) \\ &= -\ln(\sqrt{t}) + C(x) = -\ln(\sqrt{x^2 + y^2}) + C(x) \end{aligned}$$

Remember that the “constant of integration”  $C$  actually depends on the value of  $x$  that we fixed, that is why we obtain  $C(x)$ . To determine this function  $C(x)$  we

have to resort to the other Cauchy-Riemann equation:  $v_x = -u_y$ , where now we can compute  $v_x$ :

$$\begin{aligned}v_x &= (-\ln(\sqrt{x^2 + y^2}) + C(x))_x = -\frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x + C'(x) \\ &= -\frac{x}{x^2 + y^2} + C'(x).\end{aligned}$$

If we now compare this last expression with the value of  $-u_y$  (computed above) we see that  $C'(x) = 0$ , so that  $C(x)$  is a proper constant.

Conclusion:  $v = -\ln(\sqrt{x^2 + y^2}) + C$  for any constant  $C$ .

(d) Check that, up to an additive constant,  $v = \text{Im}(f)$ .

It is clear that, except for the constant  $C$ ,  $v$  as computed above coincides with  $\text{Im}(f)$ .