Trigonometric Polynomials

First define the first few basic elements

```
> a[0]:=t->1/sqrt(2):
> a[1]:=t->cos(t):
> b[1]:=t->sin(t):
> a[2]:=t->cos(2*t):
> b[2]:=t->sin(2*t):
> a[3]:=t->cos(3*t):
> b[3]:=t->sin(3*t):
```

Define a test function:

```
> ff:=t->abs(t):
```

The inner product

```
> IP:=(f,g)->(1/Pi)*int(f(u)*g(u),u=-Pi..Pi);
```

$$IP := (f, g) \to \frac{1}{\pi} \int_{-\pi}^{\pi} f(u)\, g(u)\, du$$

For example, the inner product of ff and a[0] is:

```
> IP(ff,a[0]);
```

$$\frac{\pi\sqrt{2}}{2}$$

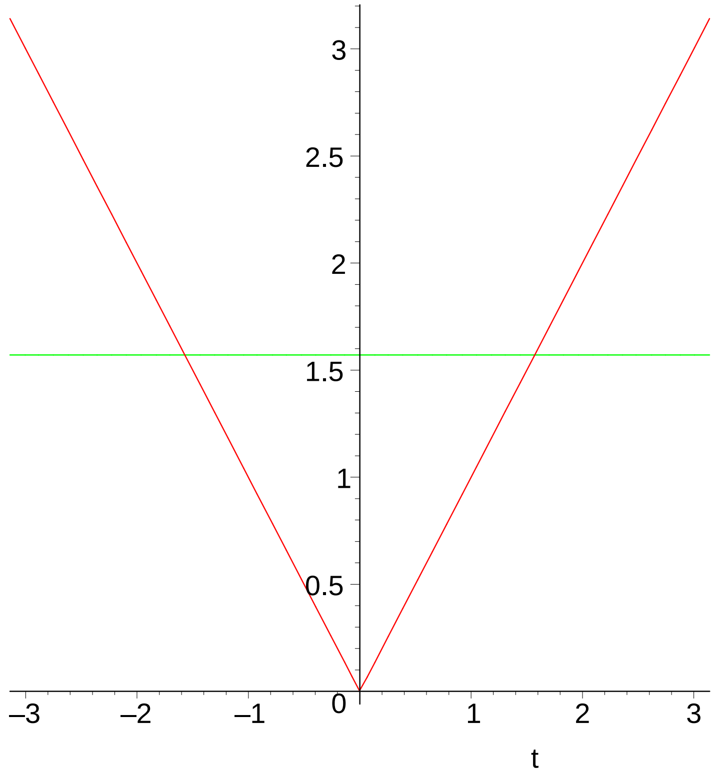
Find the orthogonal projection of ff on the subspace <a[0]>:

```
> ff0:=t->(IP(ff,a[0])*a[0](t));
```

$$ff0 := t \to \mathrm{IP}(ff, a_0)\, a_0(t)$$

We can see the graph of both ff and its projection on <a[0]>:

```
> plot([ff(t),ff0(t)],t=-Pi..Pi);
```



Next we consider the orthogonal projection on <a[0],a[1],b[1]>. The two (new) coefficients are

```
> IP(ff,a[1]);evalf(%);
```

$$-\frac{4}{\pi}$$

$$-1.273239544$$

and

```
> IP(ff,b[1]);evalf(%);
```

$$0$$

$$0.$$

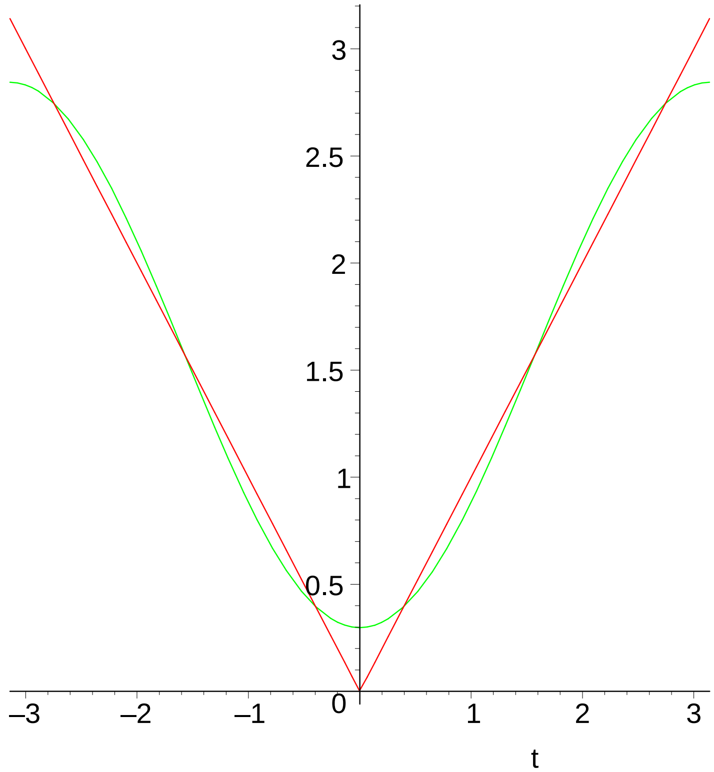
Then the projection is

```
> ff1:=t->IP(ff,a[0])*a[0](t) + IP(ff,a[1])*a[1](t) +
  IP(ff,b[1])*b[1](t);
```

$$ff1 := t \to \mathrm{IP}(ff, a_0)\, a_0(t) + \mathrm{IP}(ff, a_1)\, a_1(t) + \mathrm{IP}(ff, b_1)\, b_1(t)$$

The following plot shows ff and the projection

```
> plot([ff(t),ff1(t)],t=-Pi..Pi);
```



Next we consider the orthogonal projection on <a[0],a[1,],b[1],a[2],b[2]>. The new coefficients are:

```
> IP(ff,a[2]);evalf(%);
```

$$0$$

$$0.$$

and

```
> IP(ff,b[2]);evalf(%);
```

$$0$$

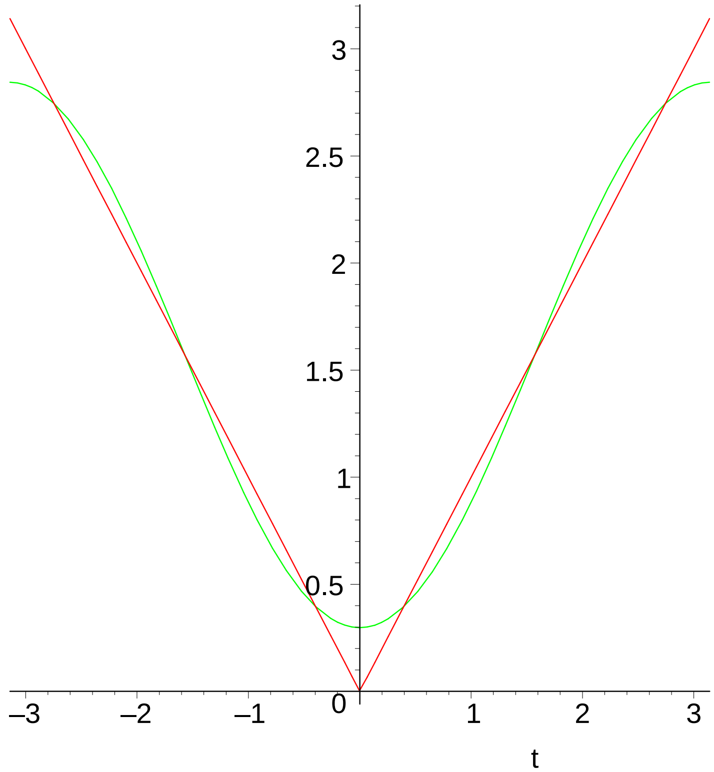
$$0.$$

Since we see that both coefficients vanish we know that the projection of ff on this subspace is the same vector as the last projection.

```
> ff2:=t->IP(ff,a[0])*a[0](t) + IP(ff,a[1])*a[1](t) +
  IP(ff,b[1])*b[1](t) + IP(ff,a[2])*a[2](t) + IP(ff,b[2])*b[2](t);
```

$$ff2 := t \to \mathrm{IP}(ff, a_0)\, a_0(t) + \mathrm{IP}(ff, a_1)\, a_1(t) + \mathrm{IP}(ff, b_1)\, b_1(t) + \mathrm{IP}(ff, a_2)\, a_2(t) + \mathrm{IP}(ff, b_2)\, b_2(t)$$

```
> plot([ff(t),ff2(t)],t=-Pi..Pi);
```



Finally, we consider the projection on <a[0],a[1],b[1],a[2],b[2],a[3],b[3]>. The new coefficients are

```
> IP(ff,a[3]);evalf(%);
```

$$-\frac{4}{9\pi}$$

$$-0.1414710605$$

```
> IP(ff,b[3]);evalf(%);
```

$$0$$

$$0.$$

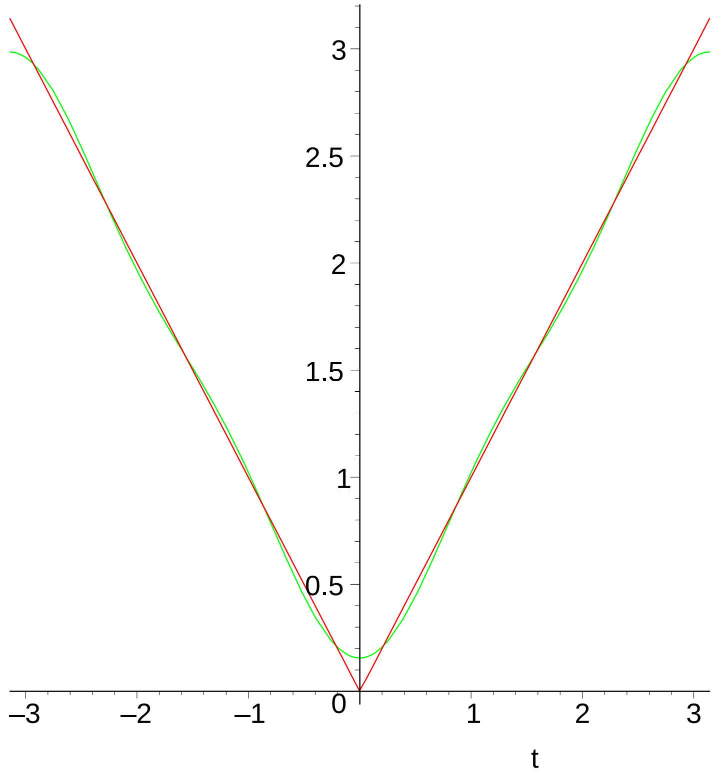
and the projection is

```
> ff3:=t->IP(ff,a[0])*a[0](t) + IP(ff,a[1])*a[1](t) +
  IP(ff,b[1])*b[1](t) + IP(ff,a[2])*a[2](t) + IP(ff,b[2])*b[2](t) +
  IP(ff,a[3])*a[3](t) + IP(ff,b[3])*b[3](t);
```

$$ff3 := t \to \mathrm{IP}(ff, a_0)\, a_0(t) + \mathrm{IP}(ff, a_1)\, a_1(t) + \mathrm{IP}(ff, b_1)\, b_1(t) + \mathrm{IP}(ff, a_2)\, a_2(t) + \mathrm{IP}(ff, b_2)\, b_2(t)$$
$$+ \mathrm{IP}(ff, a_3)\, a_3(t) + \mathrm{IP}(ff, b_3)\, b_3(t)$$

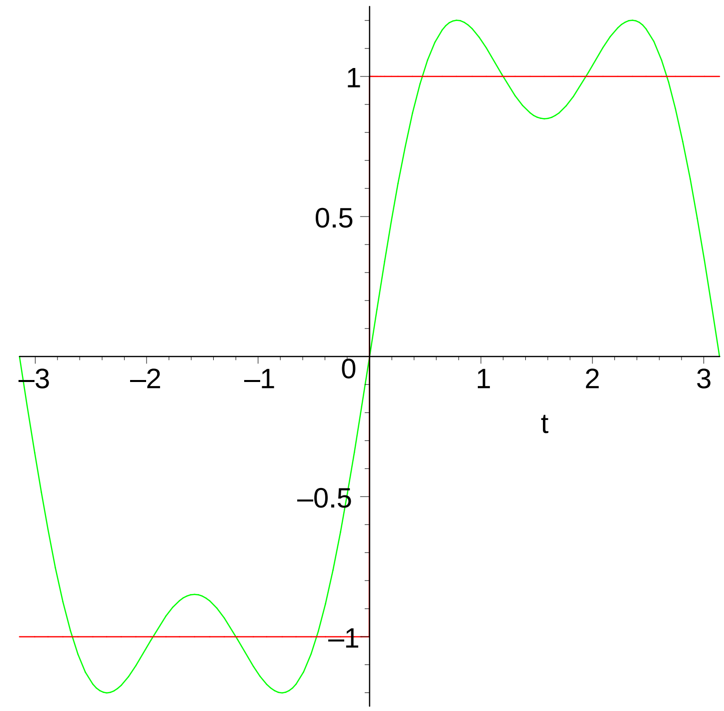
with graph

```
> plot([ff(t),ff3(t)],t=-Pi..Pi);
```



We know that ff is not differentiable at the origin whereas the projections (being trigonometric polynomials) are. How well does the derivative of the last projection approximate the derivatives near the origin, and what happens at the origin?

```
> plot([diff(ff(t),t),diff(ff3(t),t)],t=-Pi..Pi);
```



We see on the previous graph that the approximation seems to be correct except at the origin (to be expected by the nondifferentiability of ff) and at the endpoints of the interval: why would this be? The problem is that, on the one hand the trigonometric functions are periodic with period 2Pi (hence, their values at -Pi and Pi coincide). On the other hand, the derivative of ff is not a periodic function so the projection cannot approximate very well near the endpoints where the "deffect" becomes evident. There is a nice theory of harmonic analysis that explains all this behavior, and lots more!