Homework 8 - *Solution*

MATH 1100-2 - Spring 2002

1. **12.1.1.** If $f'(x) = 4x^3$, in order to find f, we have to increase the power of x and also divide by the increased power, that is, a possibility is $f(x) = x^4$. In fact, all the possible solutions are $f(x) = x^4 + C$ for some constant C.

2. **12.1.3.** If $f'(x) = x^6$, in order to find f, we have to increase the power of x and also divide by the increased power, that is, a possibility is $f(x) = \frac{x^7}{7}$. In fact, all the possible solutions are $f(x) = \frac{x^7}{7} + C$ for some constant C.

3. **12.1.5.** Using the rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ we have: $\int x^7 dx = \frac{x^8}{8} + C$.

   Verification: $(\frac{x^8}{8} + C)' = (\frac{x^8}{8})' + 0 = \frac{1}{8} \cdot 8 \cdot x^7 = x^7$.

4. **12.1.11.** We first split and then use the rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C$:

$$\int (3 - x^{3/2})dx = \int 3dx - \int x^{3/2}dx = 3\int dx - \frac{x^{5/2}}{5/2} = 3x - \frac{2x^{5/2}}{5} + C.$$

   Verification: $(3x - \frac{2x^{5/2}}{5} + C)' = (3x)' - (\frac{2x^{5/2}}{5})' + 0 = 3 - \frac{2}{5} \cdot \frac{5}{2}x^{3/2}$.

5. **12.1.15.** We first split and then use the rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C$:

$$\int (2 + 2\sqrt{x})dx = \int 2dx + \int 2\sqrt{x}dx = 2\int dx + 2\int x^{1/2}dx = 2x + 2 \cdot \frac{x^{3/2}}{3/2} + C$$
$$= 2x + \frac{4}{3}x^{3/2} + C.$$

   Verification: $(2x + \frac{4}{3}x^{3/2} + C)' = (2x)' + (\frac{4}{3}x^{3/2})' + 0 = 2 + \frac{4}{3} \cdot \frac{3}{2}x^{1/2} = 2 + 2x^{1/2}$.

6. **12.1.27.**

$$\int (x+5)^2 x dx = \int (x^2 + 10x + 25)x dx = \int (x^3 + 10x^2 + 25x)dx$$
$$= \frac{x^4}{4} + \frac{10x^3}{3} + \frac{25x^2}{2} + C.$$

7. **12.1.43.** Since $MR(x) = R'(x)$, we have $R(x) = \int MR(x)dx = \int (3x + 1)dx = 3\int xdx + \int dx = \frac{3}{2}x^2 + x + C$. But, we know that $R(0) = 0$, so that $0 = R(0) = 0 + 0 + C$ and $C = 0$. Therefore $R(x) = \frac{3}{2}x^2 + x$.

   The revenue produced by $x = 50$ units is computed by $R(50) = \frac{3}{2}(50)^2 + 50 = 3800$.

8. **12.2.1.** Using the substitution $u = x^2 + 3$, so that $du = 2xdx$ we have

$$\int (x^2 + 3)^3 2x dx = \int u^3 du = \frac{u^4}{4} + C = \frac{(x^2+3)^4}{4} + C.$$

Verification:

$$\frac{d}{dx}(\frac{(x^2+3)^4}{4}+C)=\frac{d}{dx}(\underbrace{\frac{(x^2+3)^4}{4}}_{u=x^2+3})+0=\frac{d\frac{u^4}{4}}{du}\cdot\frac{d(x^2+3)}{dx}=u^3\cdot 2x=(x^2+3)^3 2x.$$

9. **12.2.7.** We use the substitution $u=x^2+5$ so that $du=2xdx$. We see that we have to multiply by $\frac{2}{2}$:

$$\int(x^2+5)^3xdx=\int(x^2+5)^3\frac{2}{2}xdx=\int\frac{1}{2}u^3du=\frac{1}{2}\int u^3du=\frac{1}{2}\cdot\frac{u^4}{4}+C=\frac{1}{8}\cdot(x^2+5)^4+C.$$

Verification:

$$\frac{d}{dx}(\frac{1}{8}\cdot(x^2+5)^4+C)=\frac{1}{8}\frac{d}{dx}(\underbrace{(x^2+5)^4}_{v=x^2+5})+0=\frac{1}{8}\frac{dv^4}{dv}\cdot\frac{d(x^2+5)}{dx}=\frac{1}{8}\cdot 4v^3\cdot 2x=(x^2+5)^3x.$$

10. **12.2.9.** We use the substitution $u=4x-1$ so that $du=4dx$. We see that we have to multiply by $\frac{4}{4}$:

$$\int 7(4x-1)^6dx=7\int(4x-1)^6\frac{4}{4}dx=\frac{7}{4}\int u^6du=\frac{7}{4}\cdot\frac{u^7}{7}+C=\frac{(4x-1)^7}{4}+C.$$

Verification:

$$\frac{d}{dx}(\frac{(4x-1)^7}{4}+C)=\frac{1}{4}\frac{d}{dx}(\underbrace{(4x-1)^7}_{v=4x-1})+0=\frac{1}{4}\frac{dv^7}{dv}\cdot\frac{d(4x-1)}{dx}=\frac{1}{4}7v^6\cdot 4=7(4x-1)^6.$$

11. **12.2.17.** We use the substitution $u=x^4+6$ so that $du=4x^3dx$. We see that we have to multiply by $\frac{4}{4}$:

$$\int 7x^3\sqrt{x^4+6}dx=7\int x^3(x^4+6)^{1/2}\frac{4}{4}dx=\frac{7}{4}\int u^{1/2}du=\frac{7}{4}\cdot\frac{u^{3/2}}{3/2}+C=\frac{7}{6}(x^4+6)^{3/2}+C.$$

Verification:

$$\frac{d}{dx}(\frac{7}{6}(x^4+6)^{3/2}+C)=\frac{7}{6}\frac{d}{dx}(\underbrace{(x^4+6)^{3/2}}_{v=x^4+6})+0=\frac{7}{6}\frac{dv^{3/2}}{dv}\cdot\frac{d(x^4+6)}{dx}=\frac{7}{6}\cdot\frac{3}{2}\cdot v^{1/2}\cdot 4x^3$$
$$=7(x^4+6)^{1/2}x^3.$$

12. **12.2.27.** We use the substitution $u=2x^5-5$ so that $du=10x^4dx$. We see that we have to multiply by $\frac{10}{10}$:

$$\int\frac{3x^4dx}{(2x^5-5)^4}=3\int\frac{10}{10}\cdot\frac{x^4dx}{(2x^5-5)^4}=\frac{3}{10}\int\frac{du}{u^4}=\frac{3}{10}\int u^{-4}du=\frac{3}{10}\cdot\frac{u^{-3}}{-3}+C$$
$$=\frac{3}{10}\cdot\frac{(2x^5-5)^{-3}}{-3}+C=-\frac{1}{10}(2x^5-5)^{-3}+C.$$

Verification:

$$\frac{d}{dx}(-\frac{1}{10}(2x^5-5)^{-3}+C) = -\frac{1}{10}\frac{d}{dx}(\underbrace{(2x^5-5)^{-3}}_{v=2x^5-5}) + 0 = -\frac{1}{10}\frac{dv^{-3}}{dv}\cdot\frac{d(2x^5-5)}{dx}$$
$$= -\frac{1}{10}\cdot(-3)v^{-4}\cdot 10x^4 = 3(2x^5-5)^{-4}x^4.$$

13. **12.2.39.** Given the marginal revenue $MR(x) = \frac{-30}{(2x+1)^2}+30$, we find the total revenue by integration. We will use the substitution $u = 2x+1$, with $du = 2dx$:

$$R(x) = \int MR(x)dx = \int(\frac{-30}{(2x+1)^2}+30)dx = \int\frac{-30}{(2x+1)^2}\frac{2}{2}dx + \int 30dx$$
$$= -15\int u^{-2}du + 30x = -15\frac{u^{-1}}{-1} + 30x + C = 15(2x+1)^{-1} + 30x + C.$$

To determine C we impose the condition $R(0) = 0$, so that $0 = R(0) = 15(2\cdot 0+1)^{-1} + 30\cdot 0 + C = 15\cdot 1 + C = 15 + C$, so that $C = -15$. Thus, the total revenue function is $R(x) = 15(2x+1)^{-1} + 30x - 15$.

14. **12.3.1.** We use the substitution $u = x^3+4$, which leads to $du = 3x^2dx$:

$$\int\frac{3x^2}{x^3+4}dx = \int\frac{1}{u}du = \ln(|u|) + C = \ln(|x^3+4|) + C.$$

15. **12.3.3.** We use the substitution $u = 4z+1$, which leads to $du = 4dz$. Along the way, we multiply the integrand by $\frac{4}{4}$:

$$\int\frac{dz}{4z+1} = \int\frac{4}{4}\cdot\frac{dz}{4z+1} = \frac{1}{4}\int\frac{du}{u} = \frac{1}{4}\ln(|u|) + C = \frac{1}{4}\ln(|4z+1|) + C.$$

16. **12.3.17.** We use the substitution $u = 3x$, which leads to $du = 3dx$:

$$\int 3e^{3x}dx = \int e^u du = e^u + C = e^{3x} + C.$$

17. **12.3.19.** We use the substitution $u = -x$, which leads to $du = -dx$. Along the way, we multiply the integrand by $\frac{-1}{-1}$:

$$\int e^{-x}dx = \int e^{-x}\cdot\frac{-1}{-1}dx = -\int e^u du = -e^u + C = -e^{-x} + C.$$

18. **12.3.31.** We first split and then use the substitutions $u = 4x$ and $v = -x/2$, which lead to $du = 4dx$ and $dv = -\frac{1}{2}dx$:

$$\int(e^{4x} - \frac{3}{e^{x/2}})dx = \int e^{4x}dx - \int 3e^{-x/2}dx = \int e^{4x}\frac{4}{4}dx - \int 3e^{-x/2}\frac{-2}{-2}dx$$
$$= \frac{1}{4}\int e^u du - 3\cdot(-2)\int e^v dv = \frac{1}{4}e^u + 6e^v + C = \frac{1}{4}e^{4x} + 6e^{-x/2} + C.$$

19. **12.4.1.** The total cost is the integral of the marginal cost:

$$C(x) = \int MC(x)dx = \int (2x + 100)dx = \int 2xdx + \int 100dx = 2\int xdx + 100\int dx$$
$$= x^2 + 100x + K.$$

To determine the constant K we use that the fixed costs are 100, that is, $100 = C(0) = 0 + 0 + K$, so that $K = 100$ and the total cost function is $C(x) = x^2 + 100x + 100$.

20. **12.4.7.** We start by finding the total revenue $R(x)$:

$$R(x) = \int MR(x)dx = \int (44 - 5x)dx = \int 44dx - \int 5xdx = 44\int dx - 5\int xdx$$
$$= 44x - \frac{5}{2}x^2 + K$$

To find K we use that $R(0) = 0$, so that $0 = R(0) = 0 - 0 + K$ and $K = 0$. Therefore $R(x) = 44x - \frac{5}{2}x^2$.

Next we find the total cost:

$$C(x) = \int MC(x)dx = \int (3x + 20)dx = \int 3xdx + \int 20dx = 3\int xdx + 20\int dx$$
$$= \frac{3}{2}x^2 + 20x + K.$$

To find K we use that $R(80) = 11400$, so that $11400 = R(80) = \frac{3}{2}80^2 + 20 \cdot 80 + K$, so that $11400 = 11200 + K$, and $K = 200$. Therefore, $C(x) = \frac{3}{2}x^2 + 20x + 200$.

The profit function is $P(x) = R(x) - C(x) = 44x - \frac{5}{2}x^2 - (\frac{3}{2}x^2 + 20x + 200) = -4x^2 + 24x - 200$.

The maximum profit is achieved when $P'(x) = 0$, that is, when $-8x + 24 = 0$, or $x = 3$ (actually, since $P''(3) = -8 < 0$, we know that $x = 3$ is a maximum).

Finally, the maximum profit is $P(3) = -164 < 0$, so that there is a loss at the optimum level.