Homework 7 - *Solution*

MATH 1100-2 - SPRING 2002

1. **10.3.1.** We start by looking for the critical values. $f(x) = x^3 - 2x^2 - 4x + 2$, so that $f'(x) = 3x^2 - 4x - 4$. To solve $f'(x) = 0$ we use the quadratic formula and find that the critical values are $x = -\frac{2}{3}$ and $x = 2$.

   To find the absolute maximum and minimum we compare the values of f at the critical values and the values at the endpoints of the interval. The highest value is the absolute maximum and the lowest the absolute minimum. From Table 1 we conclude that the maximum happens for $x = -\frac{2}{3}$ and the minimum is $x = 2$.

x	-1	$-\frac{2}{3}$	2	3
$f(x)$	3	$\frac{94}{27} \sim 3.48$	-6	-1

   Table 1: Values of f for exercise 10.3.1.

2. **10.3.3.** We proceed as in the previous exercise. $f(x) = x^3 + x^2 - x + 1$, so that $f'(x) = 3x^2 + 2x - 1$ and the critical values are $x = -1$ and $x = \frac{1}{3}$. The second of these values can be discarded because it is not in the interval under consideration. Next we produce a table of values of f at the critical values and at the endpoints of the interval. Using Table 2 we conclude that the maximum is at $x = -1$ and the minimum at $x = -2$.

x	-2	-1	0
$f(x)$	-1	2	1

   Table 2: Values of f for exercise 10.3.3.

3. **10.3.5.**

   (a) Maximizing revenue means maximizing $R = 36x - 0.01x^2$ (for $x \geq 0$). Since $R' = 36 - 0.02x$, we see that the only critical value of R is $x = 1800$. To see if that value is a local maximum or minimum we use the second derivative test: since $R'' = -0.02 < 0$, we conclude that $x = 1800$ is a local maximum. Actually, from the fact that the graph of R is a parabola pointing downwards we immediately conclude that the local minimum is actually the absolute minimum. Thus, the minimum revenue is $R(1800) = 32400$.

   (b) If the production is limited to $x \in [0, 1500]$, the absolute maximum computed above cannot be reached (it is not in the region under consideration). So we have to consider the values of R at the endpoints $x = 0$ and $x = 1500$ (there are no critical values between them). Since $R(0) = 0$ and $R(1500) = 31500$, we conclude that the maximum revenue for a production of up to $x = 1500$ is 31500.

4. **10.3.25.** Since profit $P(x) = R(x) - C(x) = 4600x - (45000 + 100x + x^3) = -x^3 + 4500x - 45000$, we look for the critical values of P by solving $P'(x) = 0$. Since $P'(x) = -3x^2 + 4500$, the critical values occur at $x = \pm\sqrt{1500} \simeq \pm 38.73$. x being a

number of units, it has to be positive, so the only interesting critical value happens at $x \simeq 38.73$. The second derivative test says that since $P''(38.73) = -0.02 < 0$, the point is a local maximum.

But, $x = 38.73$ is not an integer! So we check the two closest integers to see which one has the higher value: $P(38) = 1353.56$ and $P(39) = 1388.79$. Then we conclude that the maximum profit is 1388.79, which happens when the production level is $x = 39$.

5. **10.4.1.**

   (a) For each district we have to find first the critical values of the sales function. Since $S_1 = 30 + 20x_1 - 0.4x_1^2$ and $S_2 = 20 + 36x_2 - 1.3x_2^2$, we have $S_1' = 20 - 0.8x_1$ and $S_2' = 36 - 2.6x_2$, we conclude that the critical values are $x_1 = 25$ and $x_2 = 13.8462$. Since we have $S_1'' = -0.8 < 0$, and $S_2'' = -2.6$, we conclude (using the second derivative test) that both critical values are local maxima. Therefore, the sales revenue are maximized by spending $x_1 = 25$ and $x_2 = 13.8462$ (millions of dollars in each case).

   (b) This is just the sum of the amounts needed to maximize each region: $38.8462 = 25 + 13.8462$ millions of dollars.

6. **10.4.5.**

   (a) Since the hourly number of units produced after t hours is given by $y(t) = 70t + \frac{1}{2}t^2 - t^3$, what we want to do is to maximize this function on the interval $[0, 8]$. We start by finding the critical values, that is, solving $R'(t) = 0$. Since $R' = 70 + t - 3t^2$, we see that the critical points would be $t = -\frac{14}{3}$ and $t = 5$. But, the first is not in the region under consideration. Therefore, the only critical point is $t = 5$. From Table 3 we can compare the values of y at the critical point and at the two endpoints. We conclude that the maximum of y happens after $t = 5$ hours.

t	0	5	8
$y(t)$	0	237.5	80

Table 3: Values of y for exercise 10.4.5.

   (b) That is $y(5) = 237.5$, reading from Table 3.

7. **10.4.7.** That is, we want to find the price p that maximizes the expenditure function $E(p) = 10000p - 100p^2$. We look for the critical points: $E' = 10000 - 200p$, so that the critical value is $p = 50$. Since $E'' = -200 < 0$, by the second derivative test we conclude that 50 is a local maximum. Actually, since the graph of E is that of a parabola pointing down we conclude that 50 the maximum of E is achieved when the price is $p = 50$.

8. **10.4.13.** We have to find the maximum of the function $p(t) = \frac{6.4t}{t^2+64} + 0.05$ (for $t \geq 0$). We start by looking for the critical values.

$$p'(t) = \frac{6.4(t^2 + 64) - 6.4t \cdot 2t}{(t^2 + 64)^2} = \frac{-6.4t^2 + 409.6}{(t^2 + 64)^2}.$$

Then, the critical values satisfy: $-6.4t^2 + 409.6 = 0$, and they are $t = \pm 8$. Clearly, $t = -8$ is not a possibility in the context of this problem, and the only critical value is $t = 8$. We can use the first derivative test to conclude that $t = 8$ is a local maximum. Then, the maximum value of p happens when $t = 8$ and that value is $p(8) = 0.45$.

9. **10.4.31.** The cost function that we want to minimize is composed of two parts: plate preparation cost $(2 \cdot x)$, and press use cost, $12.5 \cdot h$, where h is the number of press-hours needed to complete the job. Thus, in principle, the cost is

$$C = 2x + 12.5h.$$

There is a relation between x and h: if x plates are used and 1000 press impressions can be made per hour, in order to produce 100000 posters, $\frac{100000}{1000x}$ press hours are needed. That is:

$$h = \frac{100000}{1000x}.$$

All together, the cost function becomes:

$$C = 2x + 12.5 \cdot \frac{100}{x} = 2x + \frac{1250}{x}.$$

Now, we want to find the critical values of C. $C' = 2 - \frac{1250}{x^2}$, so that $C' = 0$ means that $2 - \frac{1250}{x^2} = 0$, which says that $x = \pm\sqrt{625} = \pm 25$. Since x cannot be negative, the only interesting critical value is $x = 25$.

We use the second derivative test to check if $x = 25$ is a local minimum: $C'' = (2 - \frac{1250}{x^2})' = \frac{2500}{x^3}$. Then, $C''(25) = 0.16 > 0$,and $x = 25$ is a local minimum, as we wanted.

10. **10.5.1.**

(a) From the graph, there is a vertical asymptote at $x = 2$. From the formula $f(x) = \frac{x-4}{x-2}$, we see that the denominator vanishes at $x = 2$, whereas the numerator doesn't, thus producing a vertical asymptote when $x = 2$.

(b) From the graph it looks like $\lim_{x\to+\infty} f(x) = 1$. From the formula we compute

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x-4}{x-2} = \lim_{x\to+\infty} \frac{\frac{x-4}{x}}{\frac{x-2}{x}} = \lim_{x\to+\infty} \frac{1-\frac{4}{x}}{1-\frac{2}{x}} = \frac{1-0}{1-0} = 1.$$

Notice that the previous computation also computes $\lim_{x\to-\infty} f(x) = 1$.

(c) From the graph, it looks like there is a horizontal asymptote with $y = 1$. From the formula, the previous item shows that this is, in fact, the case.

(d) From the graph it looks like $\lim_{x\to-\infty} f(x) = 1$. As we noted above this is also the result of the computation using the formula.

11. **10.5.5.** For $y = \frac{2x}{x-3}$, we see that the denominator vanishes when $x = 3$. Since the numerator doesn't vanish at $x = 3$, we conclude that $x = 3$ is a vertical asymptote.

To detect horizontal asymptotes we have to compute:

$$\lim_{x\to\infty} \frac{2x}{x-3} = \lim_{x\to\infty} \frac{\frac{2x}{x}}{\frac{x-3}{x}} = \lim_{x\to\infty} \frac{2}{1-\frac{3}{x}} = \frac{2}{1-0} = 2.$$

Then, there is a horizontal asymptote at $y = 2$.

12. **10.5.9.** For $y = \frac{3x^3-6}{x^2+4}$, we look for zeros of the denominator x^2+4. But this is always a positive quantity that can't vanish. Therefore, there are no vertical asymptotes.

To detect horizontal asymptotes we have to compute:

$$\lim_{x\to\infty} \frac{3x^3-6}{x^2+4} = \lim_{x\to\infty} \frac{\frac{3x^3-6}{x^2}}{\frac{x^2+4}{x^2}} = \lim_{x\to\infty} \frac{3x-\frac{6}{x^2}}{1+\frac{4}{x^2}} = \infty.$$

Therefore, there is no horizontal asymptote.