Homework 6 - *Solution*

MATH 1100-2 - SPRING 2002

1. **11.4.1.** For $y = x^3 - 3x$, we start by taking $\frac{d}{dt}$ on both sides:

$$\frac{dy}{dt} = \frac{d}{dt}(x^3 - 3x) = \frac{d(x^3)}{dt} - \frac{d(3x)}{dt} = 3x^2\frac{dx}{dt} - 3\frac{dx}{dt}.$$

We now substitute $x = 2$ and $\frac{dx}{dt} = 4$ and obtain $\frac{dy}{dt} = 3 \cdot 2^2 \cdot 4 - 3 \cdot 4 = 48 - 12 = 36$.

2. **11.4.17.** Both P and x are functions of time (t). We start by applying $\frac{d}{dt}$ to both sides of $P = 180x - \frac{x^2}{1000} - 2000$:

$$\frac{dP}{dt} = \frac{d}{dt}(180x - \frac{x^2}{1000} - 2000) = 180\frac{dx}{dt} - \frac{1}{1000}\frac{dx^2}{dt} - 0 = 180\frac{dx}{dt} - \frac{1}{500}x \cdot \frac{dx}{dt}.$$

Now we can replace $x = 100$ and, since x is increasing at a rate of 10 units per day, we have that $\frac{dx}{dt} = 10$.

$$\frac{dP}{dt} = 180 \cdot 10 - \frac{1}{500} \cdot 100 \cdot 10 = 1798.$$

Thus, the profit is increasing at 1798 dollars per day.

3. **11.4.19.** We want to know the rate at which the price is changing. That is, we want to know $\frac{dp}{dt}$. Start by applying $\frac{d}{dt}$ to both sides of $p = \frac{1000-10x}{400-x}$:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{1000 - 10x}{400 - x} = \frac{\frac{d}{dt}(1000 - 10x) \cdot (400 - x) - (1000 - 10x)\frac{d}{dt}(400 - x)}{(400 - x)^2}$$
$$= \frac{-10(400 - x)\frac{dx}{dt} + (1000 - 10x)\frac{dx}{dt}}{(400 - x)^2} = \frac{-3000\frac{dx}{dt}}{(400 - x)^2}.$$

Now, plugging in $x = 20$ and $\frac{dx}{dt} = -20$ (notice the negative sign because the demand is decreasing), we have

$$\frac{dp}{dt} = -\frac{3000 \cdot (-20)}{(400 - 20)^2} = \frac{150}{361} \simeq 0.4155.$$

Thus, the price is increasing at approximately 42 cents per day.

4. **10.1.3.**

   (a) f' changing from positive to negative means that f changes from increasing to decreasing, which happens at $x = 1$.

   (b) f' changing from negative to positive means that f changes from decreasing to increasing, which happens at $x = 4$.

   (c) f' doesn't change sign at $x = -1$, $x = 0$ and $x = 5$ since near these points the function is increasing.

5. **10.1.5.**

(a) The critical values of f are the points where the derivative vanishes: $x = 3$ and $x = 7$.

(b) f increases where $f'(x) > 0$, that is, on $(3, 7)$.

(c) f decreases where $f'(x) < 0$, that is, on $(-\infty, 3)$ and on $(7, \infty)$.

(d) At $x = 7$ f goes from increasing to decreasing, so that $x = 7$ is a relative maximum.

(e) At $x = 3$ f goes from decreasing to increasing, so that $x = 3$ is a relative minimum.

6. **10.1.7.** The critical values of $f(x) = 2x^3 - 12x^2 + 6$ are those values of x where $f'(x) = 0$ or $f'(x)$ is not defined. Being f a polynomial, $f'(x) = 6x^2 - 24x$ is always defined. So, the critical values arise as solutions of $f'(x) = 0$:

$$\begin{aligned} 6x^2 - 24x &= 0 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \end{aligned}$$

So that the critical values are $x = 0$ and $x = 4$.

7. **10.1.9.** In the previous exercise we found the critical values of f. Now we find the signs of the derivative in the corresponding intervals. We have:

$$\begin{aligned} f'(-1) &= 6 + 24 = 30 > 0 \\ f'(1) &= 6 - 24 = -18 < 0 \\ f'(5) &= 30 > 0 \end{aligned}$$

The sign diagram is shown in Figure 1.

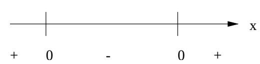

Figure 1: Sign diagram for exercise 10.1.9

8. **10.1.17.** For $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 1$ we have

(a) $y' = x^2 + x - 2$.

(b) Since the derivative is always defined, we have to solve $f'(x) = 0$, that is, $x^2 + x - 2 = 0$, which, for instance using the quadratic formula, has solutions $x = -2$ and $x = 1$. Thus, the critical values are $x = -2$ and $x = 1$.

(c) Evaluating the critical values we obtain the critical points: $(-2, \frac{13}{3})$ and $(1, -\frac{1}{6})$.

(d) Since we know the critical points, we evaluate f' at some intermediate points:

$$\begin{aligned} f'(-3) &= 4 > 0 \\ f'(0) &= -2 < 0 \\ f'(2) &= 4 > 0 \end{aligned}$$

Thus f is increasing on $(-\infty, -2)$ and $(1, \infty)$, while it is decreasing on $(-2, 1)$.

(e) Since f goes from increasing to decreasing at $x = -2$, we conclude that $x = 2$ is a relative maximum. Conversely, since f goes from decreasing to increasing at $x = 1$ we conclude that $x = 1$ is a local minimum. The graph of f can be seen on Figure 2.

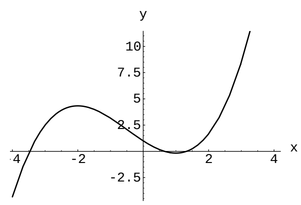

Figure 2: Graph of f for exercise 10.1.17 (e)

9. **10.1.29.** We start by computing the derivative of $f(x) = 3x^5 - 5x^3 + 1$: $f'(x) = 15x^4 - 15x^2$.

   Since the derivative is defined always (can be computed always), the critical values are the solutions of $f'(x) = 0$. That is, $15x^4 - 15x^2 = 0$, or $x^4 - x^2 = 0$. To solve this last equation we take x^2 as common factor: $x^2(x^2 - 1) = 0$. Now, if a product is 0, at least one of the two terms must vanish, so that $x = 0$ or $x^2 - 1 = 0$, which has solutions $x = 1$ and $x = -1$. Thus, the critical values are $x = -1$, $x = 0$ and $x = 1$.

   Now that we have the critical values, we can find where the function is increasing and decreasing by evaluating the derivative at some intermediate points:

$$\begin{aligned} f'(-2) &= 180 > 0 \\ f'(-\frac{1}{2}) &= -\frac{45}{16} < 0 \\ f'(\frac{1}{2}) &= -\frac{45}{16} < 0 \\ f'(2) &= 180 > 0 \end{aligned}$$

   We conclude that f is increasing on $(-\infty, -1)$ and on $(1, \infty)$, while it is decreasing on $(-1, 1)$.

   With the previous data we see that f goes from increasing to decreasing at $x = -1$, so that it is a local maximum; f is decreasing on both left and right of $x = 0$, so that $x = 0$ is an inflection point and since f goes from decreasing to increasing at $x = 1$, that point is a local minimum. These conclusions are reflected by the graph shown in Figure 3.

10. **10.2.5.** $f(x)$ is concave down to the left of c, and on (d, e).

11. **10.2.7.** $f''(x) > 0$ means that f is concave up, which happens on (c, d) and to the right of e.

12. **10.2.9.** These are points where the concavity changes: c, d and e.

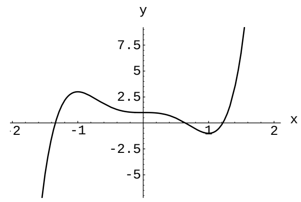

Figure 3: Graph of f for exercise 10.1.29

13. **10.2.19.** We start by finding the first derivative of $y = x^4 - 16x^2$:

$$\frac{dy}{dx} = 4x^3 - 32x.$$

Then, the critical values are the zeroes of y', that is $4x^3 - 32x = 0$, or $4x(x^2 - 8) = 0$. Then, the critical values are $x = 0$ and $x = \pm\sqrt{8}$. We want to use the second derivative test to decide what kind of points are these.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - 32x) = 12x^2 - 32.$$

Then,

$$\begin{aligned} y''(-\sqrt{8}) &= 64 > 0 \\ y''(0) &= -32 < 0 \\ y''(\sqrt{8}) &= 64 > 0 \end{aligned}$$

Thus, $x = 0$ is a local maximum and $x = \pm\sqrt{8}$ are local minima. Our conclusions are shown in Figure 4.

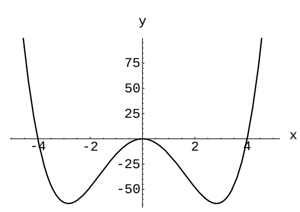

Figure 4: Graph of f for exercise 10.2.19

14. **10.2.37.**

(a) Production is maximized when we find a maximum of $P(t) = 27t + 12t^2 - t^3$. We start by finding $\frac{dP}{dt} = 27 + 24t - 3t^2$.
To find the maximum, we start by looking for critical values: $\frac{dP}{dt} = 0$ so that $27 + 24t - 3t^2 = 0$. Using the quadratic formula (or any other method), we find that $t = -1$ and $t = 9$. Clearly, none of the solutions is acceptable. But then, if we evaluate the derivative at any point between -1 and 9 (for instance $t = 0$) we

see that $\frac{dP}{dt} > 0$, so that the production is increasing all the time: the maximum production will be at the end of the 8 hour shift (which makes intuitive sense, unless the worker destroys part of his work during the day!).

(b) To maximize the rate of production, we want to maximize the function $R(t) = \frac{dP}{dt} = 27 + 24t - 3t^2$. We start by computing its derivative: $R'(t) = 24 - 6t$. The critical points are the solutions of $24-6t = 0$, that is $t = 4$. Since $R''(t) = -6 < 0$, the second derivative test says that $t = 4$ is a local maximum of R, that is, of the production rate.