Homework 3 - *Solution*

MATH 1100-2 - SPRING 2002

1. **9.4.1.** $y = 4 \Rightarrow y' = (4)' = 0$. (The derivative of a constant function is 0.)

2. **9.4.3.** $y = x \Rightarrow y' = (x)' = 1$. (This is the derivative of a power.)

3. **9.4.5.** $f(x) = 2x^3 - x^5$, so
$$f'(x) = (2x^3 - x^5)' = (2x^3)' - (x^5)' = 2(x^3)' - 5x^4 = 2 \cdot 3x^2 - 5x^4 = 6x^2 - 5x^4.$$

4. **9.4.7.** $y = 6x^4 - 5x^2 + x - 2$, so:
$$\begin{aligned} y' &= (6x^4 - 5x^2 + x - 2)' = (6x^4)' - (5x^2)' + (x)' - (2)' = 6(x^4)' - 5(x^2)' + 1 - 0 \\ &= 6 \cdot 4x^3 - 5 \cdot 2x + 1 = 24x^3 - 10x + 1. \end{aligned}$$

5. **9.4.15.** $y = x^{-5} + x^{-8} - 3$, so:
$$y' = (x^{-5}+x^{-8}-3)' = (x^{-5})'+(x^{-8})'-(3)' = -5x^{-5-1}+(-8)x^{-8-1}-0 = -5x^{-6}-8x^{-9}.$$

6. **9.4.17.** $y = 3x^{11/3} - 2x^{7/4} - x^{1/2} + 8$, so:
$$\begin{aligned} y' &= (3x^{11/3} - 2x^{7/4} - x^{1/2} + 8)' = (3x^{11/3})' - (2x^{7/4})' - (x^{1/2})' + (8)' \\ &= 3(x^{11/3})' - 2(x^{7/4})' - (x^{1/2})' + 0 = 3 \cdot \frac{11}{3}x^{11/3-1} - 2 \cdot \frac{7}{4}x^{7/4-1} - \frac{1}{2}x^{1/2-1} \\ &= 11x^{8/3} - \frac{7}{2}x^{3/4} - \frac{1}{2}x^{-1/2}. \end{aligned}$$

7. **9.4.23.** The equation of the tangent line to the graph of $f(x)$ at x_0 is $y = f'(x_0)(x - x_0) + f(x_0)$, or in terms of $y(x)$, $y = y'(x_0)(x - x_0) + y(x_0)$.

   We start by computing $y'(x)$.
$$y' = (x^3 - 3x^2 + 5)' = (x^3)' - (3x^2)' + (5)' = 3x^2 - 3(x^2)' + 0 = 3x^2 - 6x.$$
   Then, $y'(1) = 3 \cdot 1^2 - 6 \cdot 1 = -3$, and $y(1) = 1^3 - 3 \cdot 1^2 + 5 = 3$.

   All together, the equation of the tangent line at $x_0 = 1$ is $y = -3(x-1)+3 = -3x+6$.

8. **9.5.1.** $y = (x+3)(x^2 - 2x)$. We use the product rule and obtain:
$$\begin{aligned} y' &= ((x+3)(x^2-2x))' = (x+3)' \cdot (x^2 - 2x) + (x+3) \cdot (x^2 - 2x)' \\ &= ((x)' + (3)') \cdot (x^2 - 2x) + (x+3) \cdot ((x^2)' - (2x)') \\ &= (1+0) \cdot (x^2 - 2x) + (x+3) \cdot (2x - 2) = (x^2 - 2x) + (x+3) \cdot (2x-2). \end{aligned}$$

9. **9.5.5.** $f(x) = (x^{12} + 3x^4 + 4)(4x^3 - 1)$. We use the product rule and obtain:
$$\begin{aligned} f'(x) &= ((x^{12} + 3x^4 + 4)(4x^3 - 1))' \\ &= (x^{12} + 3x^4 + 4)' \cdot (4x^3 - 1) + (x^{12} + 3x^4 + 4) \cdot (4x^3 - 1)' \\ &= ((x^{12})' + (3x^4)' + (4)') \cdot (4x^3 - 1) + (x^{12} + 3x^4 + 4) \cdot ((4x^3)' - (1)') \\ &= (12x^{11} + 3(x^4)' + 0) \cdot (4x^3 - 1) + (x^{12} + 3x^4 + 4) \cdot (4(x^3)' - 0) \\ &= (12x^{11} + 12x^3) \cdot (4x^3 - 1) + (x^{12} + 3x^4 + 4) \cdot 12x^2. \end{aligned}$$

10. **9.5.9.** $y = (x^2+x+1)(\sqrt[3]{x} - 2\sqrt{x} + 5) = (x^2+x+1)(x^{1/3} - 2x^{1/2} + 5)$. We use the product rule and obtain:

$$\begin{aligned}
y' &= ((x^2+x+1)(x^{1/3} - 2x^{1/2} + 5))' \\
&= (x^2+x+1)' \cdot (x^{1/3} - 2x^{1/2} + 5) + (x^2+x+1) \cdot (x^{1/3} - 2x^{1/2} + 5)' \\
&= ((x^2)' + (x)' + (1)') \cdot (x^{1/3} - 2x^{1/2} + 5) + (x^2+x+1) \cdot ((x^{1/3})' - (2x^{1/2})' + (5)') \\
&= (2x+1+0) \cdot (x^{1/3} - 2x^{1/2} + 5) + (x^2+x+1) \cdot (\frac{1}{3}x^{-2/3} - 2(x^{1/2})' + 0) \\
&= (2x+1) \cdot (x^{1/3} - 2x^{1/2} + 5) + (x^2+x+1) \cdot (\frac{1}{3}x^{-2/3} - 2 \cdot \frac{1}{2}x^{-1/2})
\end{aligned}$$

11. **9.5.13.** $y = \frac{x}{x^2-1}$. Using the quotient rule we obtain:

$$\begin{aligned}
y' = (\frac{x}{x^2-1})' &= \frac{(x)'(x^2-1) - x(x^2-1)'}{(x^2-1)^2} = \frac{1 \cdot (x^2-1) - x((x^2)' - (1)')}{(x^2-1)^2} \\
&= \frac{x^2 - 1 - x(2x-0)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2}.
\end{aligned}$$

12. **9.5.25.**

    (a) The slope of the tangent line at $x = 2$ is the derivative $y'(2)$. We start by computing the derivative $y'(x)$:

$$\begin{aligned}
y' = (\frac{x^2+1}{x+3})' &= \frac{(x^2+1)'(x+3) + (x^2+1)(x+3)'}{(x+3)^2} \\
&= \frac{((x^2)' + (1)')(x+3) + (x^2+1)((x)' + (3)')}{(x+3)^2} = \frac{(2x+0)(x+3) + (x^2+1)(1+0)}{(x+3)^2} \\
&= \frac{2x(x+3) + x^2 + 1}{(x+3)^2} = \frac{3x^2+6x+1}{(x+3)^2}.
\end{aligned}$$

    We now plug in $x = 2$ and get the slope at the point is $y'(2) = \frac{3 \cdot 2^2 + 6 \cdot 2 + 1}{(2+3)^2} = \frac{25}{25} = 1$.

    (b) The instantaneous rate of change of y at $x = 2$ is precisely the derivative $y'(2) = 1$, as we saw above.

13. **9.6.5.** We think of $f(x) = \frac{1}{(x^2+2)^3}$ as $f(x) = g(h(x))$, where $g(x) = \frac{1}{x^3} = x^{-3}$ and $h(x) = x^2+2$. In order to find $f'(x)$ I have to use the chain rule ($f'(x) = g'(h(x)) \cdot h'(x)$), so that I need to know g' and h' first.

$$\begin{aligned}
g'(x) &= (x^{-3})' = -3x^{-4} \\
h'(x) &= (x^2+2)' = (x^2)' + (2)' = 2x + 0 = 2x
\end{aligned}$$

Now we use the chain rule:

$$f'(x) = g'(h(x)) \cdot h'(x) = -3(h(x))^{-4} \cdot 2x = -3(x^2+2)^{-4} \cdot 2x = -\frac{6x}{(x^2+2)^4}.$$

14. **9.6.15.** We think of $s(x) = 4\sqrt{3x - x^2}$ as $s(x) = g(h(x))$, where $g(x) = 4\sqrt{x} = 4x^{1/2}$ and $h(x) = 3x - x^2$. In order to find $s'(x)$ I have to use the chain rule ($s'(x) = g'(h(x)) \cdot h'(x)$), so that I need to know g' and h' first.

$$\begin{aligned} g'(x) &= (4x^{1/2})' = 4(x^{1/2})' = 4 \cdot \frac{1}{2}x^{-1/2} = \frac{2}{x^{1/2}} \\ h'(x) &= (3x - x^2)' = (3x)' - (x^2)' = 3(x)' - 2x = 3 - 2x \end{aligned}$$

Now we use the chain rule:

$$s'(x) = g'(h(x)) \cdot h'(x) = \frac{2}{(h(x))^{1/2}} \cdot (3 - 2x) = \frac{2(3 - 2x)}{\sqrt{3x - x^2}}.$$

15. **9.6.27.** The equation of the tangent line to the graph of $f(x)$ at the point x_0 is $y = f'(x_0)(x - x_0) + f(x_0)$. Thus we need to know the slope of the line, that is, the derivative of the function at x_0, $f'(x_0) = y'(x_0)$. In our case, $x_0 = 3$.

We start by finding the derivative $y'(x)$. For that, we think of y as $y(x) = g(h(x))$, for $g(x) = \sqrt{x} = x^{1/2}$ and $h(x) = 3x^2 - 2$. In order to use the chain rule, we compute

$$\begin{aligned} g'(x) &= (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \\ h'(x) &= (3x^2 - 2)' = (3x^2)' - (2)' = 3(x^2)' - 0 = 3 \cdot 2x = 6x \end{aligned}$$

Next we use the chain rule:

$$y'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{2\sqrt{h(x)}} \cdot 6x = \frac{3x}{\sqrt{3x^2 - 2}}$$

Then, the slope of the tangent line at the point $x = 3$ is $y'(3) = \frac{3 \cdot 3}{\sqrt{3 \cdot 3^2 - 2}} = \frac{9}{5}$. Last, $y(3) = \sqrt{3 \cdot 3^2 - 2} = 5$, so that the equation of the tangent line at $x = 3$ is

$$y = y'(3)(x - 3) + y(3) = \frac{9}{5}(x - 3) + 5$$