Homework 10 - *Solution*

MATH 1100-2 - SPRING 2002

1. **13.3.1.**

   (a) The area between the parabola and the line $y = 4$ is $\int_0^2 (4 - x^2)dx$ because the line is above the parabola in the given region.

   (b) We first compute $\int (4 - x^2)dx = \int 4dx - \int x^2 dx = 4x - \frac{x^3}{3}$, and then compute the definite integral:

   $$\int_0^2 (4 - x^2)dx = (4x - \frac{x^3}{3})|_0^2 = 4 \cdot 2 - \frac{2^3}{3} - 0 = \frac{16}{3}.$$

2. **13.3.4.**

   (a) The upper limit of the region is the curve $y = 6 - \frac{1}{4}x^2$ and the lower limit is $y = \sqrt{x}$, therefore the area is computed by $\int_0^4 (6 - \frac{1}{4}x^2 - \sqrt{x})dx$.

   (b) We start by computing $\int (6 - \frac{1}{4}x^2 - \sqrt{x})dx = 6x - \frac{1}{12}x^3 - \frac{2}{3}x^{\frac{3}{2}}$. Then the definite integral is:

   $$\int_0^4 (6 - \frac{1}{4}x^2 - \sqrt{x})dx = (6x - \frac{1}{12}x^3 - \frac{2}{3}x^{\frac{3}{2}})|_0^4 = 6 \cdot 4 - \frac{1}{12}4^3 - \frac{2}{3}4^{\frac{3}{2}} - 0 = \frac{40}{3}.$$

3. **13.3.7.**

   (a) To find the points of intersection we equate $x + 2 = x^2$, so that $x^2 - x - 2 = 0$ and from the quadratic formula we get $x = -1$ or $x = 2$.

   (b) Since for this region the line is above the parabola, the area is computed by $\int_{-1}^2 (x + 2 - x^2)dx$.

   (c) We first compute $\int (x + 2 - x^2)dx = \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3$, and then

   $$\begin{aligned}\int_{-1}^2 (x + 2 - x^2)dx &= (\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3)|_{-1}^2 \\ &= (\frac{1}{2}2^2 + 2 \cdot 2 - \frac{1}{3}2^3) - (\frac{1}{2}(-1)^2 + 2 \cdot (-1) - \frac{1}{3}(-1)^3) \\ &= \frac{10}{3} - (-\frac{7}{6}) = \frac{9}{2}.\end{aligned}$$

4. **13.3.9.**

   (a) To find the points of intersection we equate $x - x^2 = x^2 - 4x$, so that $2x^2 - 5x = 0$ and from the quadratic formula we get $x = 0$ or $x = \frac{5}{2}$.

   (b) According to the relative position of the parabolas (in the region of interest), the area is $\int_0^{\frac{5}{2}} (x - x^2 - (x^2 - 4x))dx$.

   (c) We first find $\int (x - x^2 - (x^2 - 4x))dx = \int (5x - 2x^2)dx = \frac{5}{2}x^2 - \frac{2}{3}x^3$, so that

   $$\int_0^{\frac{5}{2}} (x - x^2 - (x^2 - 4x))dx = (\frac{5}{2}x^2 - \frac{2}{3}x^3)|_0^{\frac{5}{2}} = (\frac{5}{2}(\frac{5}{2})^2 - \frac{2}{3}(\frac{5}{2})^3 - 0 = \frac{125}{4}.$$

5. **13.3.17.** We first find the points of intersection (of the curves): $\frac{1}{2}x^2 = x^2 - 2x$, so that $\frac{1}{2}x^2 - 2x = 0$ and, using the quadratic formula, we get $x = 0$ or $x = 4$.

   Next we need to find which curve is on top of the other for $0 \leq x \leq 4$. For that we take a point inside the region, for instance $x = 1$ and compare the values for the two parabolas. For the first we get $y(1) = \frac{1}{2}$ and for the second $y(1) = -1$, so that the first curve is higher than the second. Therefore, the area will be computed by $\int_0^4 (\frac{1}{2}x^2 - (x^2 - 2x))dx = \int_0^4 (-\frac{1}{2}x^2 + 2x)dx$.

   We first find $\int (-\frac{1}{2}x^2 + 2x)dx = -\frac{1}{6}x^3 + x^2$, so that

$$\int_0^4 (-\frac{1}{2}x^2 + 2x)dx = (-\frac{1}{6}x^3 + x^2)|_0^4 = -\frac{1}{6}(4)^3 + (4)^2 - 0 = \frac{16}{3}.$$

6. **13.3.29.** The average is computed by $\frac{1}{b-a}\int_a^b f(x)dx$, which in this case becomes $\frac{1}{1-(-1)}\int_{-1}^1 (x^3 - x)dx = \frac{1}{2}\int_{-1}^1 (x^3 - x)dx$.

   We first find that $\int (x^3 - x)dx = \frac{1}{4}x^4 - \frac{1}{2}x^2$, so that

$$\frac{1}{2}\int_{-1}^1 (x^3 - x)dx = \frac{1}{2}(\frac{1}{4}x^4 - \frac{1}{2}x^2)|_{-1}^1 = \frac{1}{2}(-\frac{1}{4} - (-\frac{1}{4})) = 0.$$

7. **13.4.3.** We are measuring time (t) in months because we have the *monthly* flow, therefore the total income during the first year (*i.e.*, 12 months) is $\int_0^{12} 24000e^{0.03t}dt$.

   To evaluate, using the substitution $u = 0.03t$ (so that $du = 0.03dt$), we compute $\int 24000e^{0.03t}dt = 24000\int e^u \frac{du}{0.03} = \frac{24000}{0.03}e^u = 800000e^{0.03t}$. Thus, the total income is

$$\int_0^{12} 24000e^{0.03t}dt = (800000e^{0.03t})|_0^{12} = 800000e^{0.03\cdot 12} - 800000e^0 = 346664.$$

8. **13.4.11.** For the given flow (time measured in years), the present value is computed by $\int_0^5 63000e^{-0.07t}dt$.

   To evaluate the integral we use the substitution $u = -0.07t$ so that $du = -0.07dt$, and $\int 63000e^{-0.07t}dt = 63000\int e^u \frac{du}{-0.07} = \frac{63000}{-0.07}e^u = -900000e^{-0.07t}$. Then, the present value is

$$\int_0^5 63000e^{-0.07t}dt = -900000e^{-0.07t}|_0^5 = -900000e^{-0.07\cdot 5} - (-900000e^0) = 265781.$$

   To compute the future value we simply multiply the present value by $e^{0.07\cdot 5} = 1.41907$, obtaining that the future value is $1.41907 \cdot 265781 = 377161$.

9. **13.5.1.** We use formula 5, with $a = 4$, so that

$$\int \frac{dx}{16 - x^2} = \frac{1}{2\cdot 4}\ln(|\frac{4+x}{4-x}|) + C.$$

10. **13.5.3.** We start by evaluating $\int \frac{dx}{x\sqrt{9+x^2}}$, for which we use formula 11 with $a = 3$:

$$\int \frac{dx}{x\sqrt{9+x^2}} = -\frac{1}{3}\ln(|\frac{3+\sqrt{9+x^2}}{x}|),$$

so that

$$\begin{aligned}\int_1^4 \frac{dx}{x\sqrt{9+x^2}} &= (-\frac{1}{3}\ln(|\frac{3+\sqrt{9+x^2}}{x}|))|_1^4 \\ &= -\frac{1}{3}\ln(|\frac{3+\sqrt{9+4^2}}{4}|) - (-\frac{1}{3}\ln(|\frac{3+\sqrt{9+1^2}}{1}|)) \\ &= -0.231049 - (-0.606149) = 0.3751.\end{aligned}$$

11. **13.5.9.** We use formula 3 with $a = 3$, so that

$$\int 3^x dx = \frac{1}{\ln(3)}3^x + C.$$

12. **13.5.16.** We want to use formula 6, but for that we have to use the substitution $u = 3x$, so that $u^2 = 9x^2$ and $du = 3dx$:

$$\begin{aligned}\int \sqrt{9x^2+4}dx = \int \sqrt{u^2+4}\frac{du}{3} &= \frac{1}{3}\cdot\frac{1}{2}(u\sqrt{4+u^2} + 4\ln(|u+\sqrt{4+u^2}|)) + C \\ &= \frac{1}{6}(3x\sqrt{4+9x^2} + 4\ln(|3x+\sqrt{4+9x^2}|)) + C.\end{aligned}$$

13. **13.5.20.** In order to solve this one we have to start by substituting $u = x^2$ so that $du = 2xdx$:

$$\int xe^{x^2}dx = \int e^u\frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

14. **13.5.31.** We want to use formula 8 with $a = \sqrt{7}$, but in order to do so we have to start by substituting $u = 2x$, so that $u^2 = 4x^2$ and $du = 2dx$:

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+7}} = \int \frac{1}{\sqrt{u^2+7}}\frac{du}{2} = \frac{1}{2}\int \frac{du}{\sqrt{u^2+7}} &= \frac{1}{2}\ln(|u+\sqrt{7+u^2}|) + C \\ &= \frac{1}{2}\ln(|2x+\sqrt{7+4x^2}|) + C.\end{aligned}$$

15. **13.6.2.** In terms of formula 17, we want to use $u = x$ and $dv = e^{-x}dx$. Then, using $y = -x$, so that $dy = -dx$, we have $v = \int dv = \int e^{-x}dx = -\int e^y dy = -e^y = -e^{-x}$. Then, integrating by parts (that is, using formula 17):

$$\int xe^{-x}dx = x\cdot(-e^{-x}) - (\int(-e^{-x})dx) = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C.$$

Notice that we used the fact that $du = dx$.

16. **13.6.3.** Since we have a logarithm and a polynomial in x, we will use $u = \ln(x)$ and $dv = x^2dx$, so that $v = \int dv = \int x^2dx = \frac{1}{3}x^3$. Then, integrating by parts:

$$\begin{aligned}\int x^2\ln(x)dx &= \ln(x)\cdot\frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}dx = \frac{x^3}{3}\cdot\ln(x) - \frac{1}{3}\int x^2dx \\ &= \frac{x^3}{3}\cdot\ln(x) - \frac{1}{3}\cdot\frac{1}{3}x^3 + C = \frac{x^3}{3}\cdot\ln(x) - \frac{1}{9}x^3 + C.\end{aligned}$$

17. **13.6.9.** We can use $u = \ln(x)$ and $dv = dx$, so that $v = x$. Then, integrating by parts we get:

$$\int \ln(x)dx = \ln(x)\cdot x - \int x\cdot\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x.$$

Then,

$$\int_1^e \ln(x)dx = (x\ln(x) - x)|_1^e = e\ln(e) - e - (1\ln(1) - 1) = e\cdot 1 - e - (1\cdot 0 - 1) = 0 - (-1) = 1.$$

18. **13.6.17.** We use $u = x^2$ and $dv = e^{-x}dx$, which, as we saw in Exercise 13.6.2 leads to $v = -e^{-x}$. Then we integrate by parts:

$$\int x^2e^{-x}dx = x^2\cdot(-e^{-x}) - \int(-e^{-x})2xdx = -x^2e^{-x} + 2\int xe^{-x}dx.$$

As we can see, the last integral still requires some work, but it is simpler than the integral we started with, so it seems to be that we are on the right track. In order to solve this last integral we integrate by parts once more with $u = x$ and $dv = e^{-x}dx$ so that $v = -e^{-x}$:

$$\int xe^{-x}dx = x\cdot(-e^{-x}) - \int(-e^{-x})dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C'.$$

All together we have:

$$\int x^2e^{-x}dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x} + C') = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C.$$

19. **13.6.21.** We choose $u = (\ln(x))^2$ (so that using the chain rule $\frac{du}{dx} = 2\ln(x)\cdot\frac{1}{x} \Rightarrow du = \frac{2}{x}\ln(x)dx$), and $dv = x^3dx$ so that $v = \int dv = \int x^3dx = \frac{1}{4}x^4$. Then, integrating by parts:

$$\int x^3(\ln(x))^2dx = (\ln(x))^2\frac{1}{4}x^4 - \int \frac{1}{4}x^4\cdot\frac{2}{x}\ln(x)dx = \frac{1}{4}x^4(\ln(x))^2 - \frac{1}{2}\int x^3\ln(x)dx.$$

We see that the last integral is nicer than the integral we started with (the logarithm is not squared now), but in order to solve this last integral we still have to integrate by parts once more. Similar to what we did before we let $u = \ln(x)$ and $dv = x^3dx$ so that $v = \frac{1}{4}x^4$. Then

$$\begin{aligned}\int x^3\ln(x)dx &= \ln(x)\frac{1}{4}x^4 - \int\frac{1}{4}x^4\cdot\frac{1}{x}dx = \frac{1}{4}x^4\ln(x) - \frac{1}{4}\int x^3dx \\ &= \frac{1}{4}x^4\ln(x) - \frac{1}{4}\cdot\frac{1}{4}x^4 + C' = \frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + C'\end{aligned}$$

Putting all together we have:

$$\begin{aligned}\int x^3(\ln(x))^2dx &= \frac{1}{4}x^4(\ln(x))^2 - \frac{1}{2}(\frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + C') \\ &= \frac{1}{4}x^4(\ln(x))^2 - \frac{1}{8}x^4\ln(x) + \frac{1}{32}x^4 + C.\end{aligned}$$