On Lie algebra extensions in a symplectic framework

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It is shown that the construction carried out by Cariñena and Ibort [J. Math. Phys. **29**, 541–545 (1988)] involving nonsymplectic actions of Lie groups gives rise to ‘‘true’’ noncentral extensions of the corresponding Lie algebras. © *1997 American Institute of Physics.* [S0022-2488(97)02407-2]

I. INTRODUCTION

Central extensions of Lie algebras appeared in Classical Mechanics some years before their importance in Quantum Field Theory were discovered: the Poisson bracket of the momentum maps associated to a symplectic but non Ad*-equivariant action of a Lie group G furnish a central extension of $Lie(G)$ (see Abraham—Marsden[1]).

More recently, Cariñena–Ibort,[2] and later Inamoto,[3] carried out a construction of Lie algebra extensions by studying descent-equations in a symplectic setting. In this way, non central extensions (for instance the Faddeev–Shatashvili[4]–Mickelsson[5] extension) are also considered in a classical context: they are related to actions of a group G that do not keep invariant the symplectic form.

In fact, they proved that under a suitable hypothesis, this method gives rise to a noncentral extension with values in a function space, but up to real-valued cochains.

The aim of this paper is to show that these real-valued cochains can be omitted, i.e., that one has a ‘‘true’’ noncentral extension. This will be done in section III.

For the sake of completeness we include a brief summary of the techniques used by Cariñena, Ibort and Inamoto in section II. We conclude with some remarks about the extensions of Lie algebras obtained by symplectic techniques.

II. A REVIEW OF THE CARIÑENA–IBORT CONSTRUCTION

In this section we consider a symplectic manifold $(\mathcal{M},\omega)$ and an action $\phi: G\times\mathcal{M}\to\mathcal{M}$ of a Lie group G on $\mathcal{M}$. This action is called *symplectic* if each one of the maps $\phi_g=\phi(g,\cdot):\mathcal{M}\to\mathcal{M}$ preserves the symplectic structure [i.e., $\phi_g^*(\omega)=\omega$]. In terms of the infinitesimal action this means that $L_{\tilde{X}_a}\omega=0$ for all $a\in Lie(G)$ where $L_{\tilde{X}_a}\omega$ is the Lie derivative of the symplectic form in the direction of the infinitesimal generator of the action $\tilde{X}_a$ given by $\tilde{X}_a(m)=(d/dt)\,\phi(\exp(ta),m)|_{t=0},\forall m\in\mathcal{M}$.

Even though symplectic actions have very interesting properties, they are not general enough to deal with all the examples coming from field theory. For this reason, it is also important to analyze actions that are not necessarily symplectic.

The action of G on $\mathcal{M}$ induces a natural action on the space of smooth functions on $\mathcal{M}$, $C^\infty(\mathcal{M})$. This action is given by $(g\cdot f)(m)=f(g^{-1}\cdot m)$. The derivative of that action produces a

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nontrivial action of $Lie(G)$ on $C^\infty(\mathscr{M})$ by $a\cdot f=-L_{\tilde{X}_a}f$ and under such action $C^\infty(\mathscr{M})$ becomes a $Lie(G)$-module. In general, $Lie(G)$ acts on all differential forms on $\mathscr{M}$ in the same way: $a\cdot\alpha=-L_{\tilde{X}_a}\alpha$.

The infinitesimal variation of ω under the action of $Lie(G)$ provides a family of 2-forms $\{\omega_a\}_{a\in Lie(G)}$, where

$$\omega_a=a\cdot\omega=-L_{\tilde{X}_a}\omega. \tag{1}$$

This family is the starting point for the descent equation considered in Ref. 2: given U open in $\mathscr{M}$ the following double complex is defined

$$\Omega(Lie(G),U)=\bigoplus_{p,q\geq 0}\Omega^{p,q}(Lie(G),U),$$

where $\Omega^{p,q}(Lie(G),U)$ is the vector space of p-cochains of $Lie(G)$ with values in q-differential forms on U.

The two operators, $d:\Omega^{p,q}(Lie(G),U)\to\Omega^{p,q+1}(Lie(G),U)$, the usual exterior differential, and $\delta:\Omega^{p,q}(Lie(G),U)\to\Omega^{p+1,q}(Lie(G),U)$, the Lie algebra cohomology operator[6] of $Lie(G)$ with values in the $Lie(G)$-module $\Omega^q(U)$ restricted to p-cochains, satisfy that $d\delta=\delta d$.

The family of 2-forms $\{\omega_a\}_{a\in Lie(G)}$ defined by (1) yields an element $\omega\in\Omega^{1,2}(Lie(G),\mathscr{M})$ and it is clear that $\delta\omega=0$. If U is a contractible open set of $\mathscr{M}$ the fact that $d\omega_a=0$ for all $a\in Lie(G)$ implies that there exists a family of 1-forms $\{\alpha_a\}_{a\in Lie(G)}$ [i.e., $\alpha\in\Omega^{1,1}(Lie(G),U)$] such that $d\alpha_a=\omega_a$ for all $a\in Lie(G)$.

Now consider $\lambda\in\Omega^{2,1}(Lie(G),U)$ given by $\lambda=\delta\alpha$. Explicitly, one has

$$\lambda(a,b)=(\delta\alpha)(a,b)=a\cdot\alpha_b-b\cdot\alpha_a-\alpha_{[a,b]}=-i_{\tilde{X}_a}\omega_b+i_{\tilde{X}_b}\omega_a+d(i_{\tilde{X}_b}\alpha_a-i_{\tilde{X}_a}\alpha_b)-\alpha_{[a,b]}.$$

The 1-forms $\lambda(a,b)$ are closed since $d\lambda=d\delta\alpha=\delta d\alpha=\delta\omega=0$. Hence, they are exact (on U). Then there exists $h\in\Omega^{2,0}(Lie(G),U)$ such that $dh=\lambda$.

Now let us assume that the symplectic form ω can be written as $\omega=\omega_i+\Delta\omega$ where ω_i is a symplectic form on $\mathscr{M}$ such that the action of G on $(\mathscr{M},\omega_i)$ is symplectic, and $\Delta\omega$ is a closed form (not necessarily nondegenerate). In this case the variation of ω under the action of G is the variation of $\Delta\omega$, $L_{\tilde{X}_a}\omega=L_{\tilde{X}_a}\Delta\omega$.

Since the action of G on $(\mathscr{M},\omega_i)$ is symplectic one has a well defined momentum map $J:\mathscr{M}\to Lie(G)^*$ given by $\langle J(m),a\rangle=J_a(m)$ where, for all $a\in Lie(G)$, $J_a:\mathscr{M}\to\mathbb{R}$ is such that $dJ_a=i_{\tilde{X}_a}\omega_i$.

Given $f\in C^\infty(\mathscr{M})$ its Hamiltonian vector field X_f satisfies the equality $df=i_{X_f}\omega$. So, $C^\infty(\mathscr{M})$ becomes a Lie algebra with the Lie bracket given by $\{f,g\}=\omega(X_f,X_g)$.

Let ΔX_a denote the vector field $\tilde{X}_a-X_{J_a}$ for all $a\in Lie(G)$.

Under the additional assumption $\omega(\Delta X_a,\Delta X_b)=0\,\forall a,b\in Lie(G)$, which is fulfilled in the relevant physical examples, Cariñena and Ibort[2] prove the following theorem.

Theorem: Let $(\mathscr{M},\omega)$ be a symplectic manifold and G a Lie group that acts on $\mathscr{M}$.

Assuming all the hypotheses stated above, one has that

$$d\{J_a,J_b\}-dJ_{[a,b]}=\lambda(a,b),$$

where λ is the previously defined 2-cocycle. After integration,

$$\{J_a,J_b\}-J_{[a,b]}=h(a,b)+c(a,b), \tag{2}$$

where $c(a,b)$ are integration constants that only depend on a and b.

This result will be improved in the next section by showing that the J_a's in (2) yield an extension of $Lie(G)$ associated to a canonical 2-cocycle.

III. THE CANONICAL 2-COCYCLE AND ITS ASSOCIATED EXTENSION

The canonical 2-cocycle on $Lie(G)$ with values in the $Lie(G)$-module $C^{\infty}(\mathscr{M})$ given by

$$\Omega(a,b)=\omega(\widetilde{X}_b,\widetilde{X}_a),$$

is naturally obtained from the action of G on $(\mathscr{M},\omega)$.

Proposition: With the same notations as above, we have

$$\{J_a,J_b\}-J_{[a,b]}=\Omega(a,b)+(\delta J)(a,b).$$

Proof:

$$\begin{aligned}\{J_a,J_b\}&=\omega(X_{J_a},X_{J_b})=\omega(\widetilde{X}_a,\widetilde{X}_b)-\omega(\widetilde{X}_a,\Delta X_b)-\omega(\Delta X_a,\widetilde{X}_b)\\&=\omega(\widetilde{X}_a,\widetilde{X}_b)+\omega(\Delta X_b,\widetilde{X}_a)-\omega(\Delta X_a,\widetilde{X}_b)\\&=\omega(\widetilde{X}_a,\widetilde{X}_b)+\Delta\omega(\widetilde{X}_b,\widetilde{X}_a)-\Delta\omega(\widetilde{X}_a,\widetilde{X}_b)=\omega(\widetilde{X}_a,\widetilde{X}_b)-2\Delta\omega(\widetilde{X}_a,\widetilde{X}_b).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\delta J)(a,b)&=a\cdot J_b-b\cdot J_a-J_{[a,b]}=-L_{\widetilde{X}_a}J_b+L_{\widetilde{X}_b}J_a-J_{[a,b]}=-dJ_b(\widetilde{X}_a)+dJ_a(\widetilde{X}_b)-J_{[a,b]}\\&=-\omega_i(\widetilde{X}_b,\widetilde{X}_a)+\omega_i(\widetilde{X}_a,\widetilde{X}_b)-J_{[a,b]}=2\omega_i(\widetilde{X}_a,\widetilde{X}_b)-J_{[a,b]}.\end{aligned}$$

So, $J_{[a,b]}=2\omega_i(\widetilde{X}_a,\widetilde{X}_b)-(\delta J)(a,b)$. Then, we conclude that

$$\begin{aligned}\{J_a,J_b\}-J_{[a,b]}&=\omega(\widetilde{X}_a,\widetilde{X}_b)-2\Delta\omega(\widetilde{X}_a,\widetilde{X}_b)-2\omega_i(\widetilde{X}_a,\widetilde{X}_b)+(\delta J)(a,b)\\&=\Omega(a,b)+(\delta J)(a,b),\end{aligned}$$

as we wanted. □

From the previous formula we can deduce the following

Proposition: If we take $\alpha_a=-i_{\Delta X_a}\omega-dJ_a,\forall a\in Lie(G)$ in the descent equation, then we can choose $h(a,b)=\Omega(a,b)$.

Proof:

$$d\alpha_a=d(-i_{\Delta X_a}\omega-dJ_a)=-di_{\Delta X_a}\omega=-di_{\widetilde{X}_a}\Delta\omega=-L_{\widetilde{X}_a}\Delta\omega=-L_{\widetilde{X}_a}\omega=\omega_a.$$

If we take $\alpha'_a=-i_{\Delta X_a}\omega$ (as in the last proof) we get $d\{J_a,J_b\}-dJ_{[a,b]}=(\delta\alpha')(a,b)$. Now we can combine the last expression with the previous proposition:

$$d\Omega(a,b)+d(\delta J)(a,b)=(\delta\alpha')(a,b),$$

so that $d\Omega(a,b)=(\delta\alpha)(a,b)$ as we wanted. □

It is worth mentioning that for cochains taking values in a function space, $J_{[a,b]}$ is not a coboundary as it is when the cochains are $\mathbb{R}$-valued.

This general framework encompasses the Mickelsson-Faddeev extension appearing in Quantum Field Theory, as it is shown by Inamoto.[3] In this case, one can see that the 2-cocycle on $Lie(\mathscr{G})$ (where $\mathscr{G}$ is the group of gauge transformations) defined by $h(a,b) = 1/24\pi \int_{S^3} tr(A[da,db])$ (see Inamoto[3] or Pressley and Segal[7]) with values in a function space, corresponds to the canonical 2-cocycle $\Omega(a,b)=\omega(X_b,X_a)$.

Remark: The hypothesis $\omega(\Delta X_a,\Delta X_b)=0$ is trivially satisfied if the action of the Lie group G is symplectic (just take $\omega=\omega_i$). In this case, the procedure described above gives rise to a central extension and it is easy to see that it corresponds to the one defined by the Poisson brackets of the momentum maps.

So, given an action of a Lie group G on a symplectic manifold $(\mathscr{M},\omega)$ satisfying $\omega(\Delta X_a,\Delta X_b)=0$ as before, the type of the resulting extension is determined by the behavior of the symplectic form ω under the action of G.

(1) If $\omega=d\theta$ (i.e., the symplectic form is exact) and the action of G leaves θ invariant, then the extension is trivial.[1]
(2) If the action of G leaves ω invariant (but not necessarily θ), then the extension is central.[1]
(3) If the action is not symplectic, the extension turns out to be noncentral in general.

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