1. Setting

Throughout the entire talk, we will be in the following setting:

$G$: finite group
$k$: field
$R$: polynomial ring over $k$ (standard grading)
$G$ acts linearly on $R$:

$$G|_k \equiv 1$$

$$G \rightarrow \text{GL}_k(R_1)$$

(We will assume that $G$ acts faithfully.)

$R^G = \{ r \in R : \forall g \in G, g(r) = r \}$ is the ring of invariants.

When is $R^G$ “good”?

**Example 1.1.** $S_n$ acts on $k[x_1, \ldots, x_n]$ (any $k$) by permuting variables. It is a classical result that

$$R^{S_n} = k[e_1, \ldots, e_n]$$

where $\{e_i\}$ are the elementary symmetric polynomials. The invariant ring is a polynomial ring, with a non-standard grading.

**Example 1.2.** $A_n$ acts on $k[x_1, \ldots, x_n]$ (any $k$) by permuting variables. Since we are asking less of an element $r$ for it to show up in this fixed ring, we expect the ring of invariants to be bigger. In fact, only one essentially new element shows up:

$$R^{A_n} = k[e_1, \ldots, e_n, \Delta]$$

where $\Delta$ is the discriminant. If $\text{char } k \neq 2$, $\Delta = \prod_{i < j} (x_i - x_j)$, and if $\text{char } k = 2$, a similar polynomial of the same degree suffices as the extra generator. The invariant ring is a hypersurface.

**Example 1.3.** $G = \langle g \rangle \cong \mathbb{Z}/3\mathbb{Z}$ acts on $R = \mathbb{C}[x, y]$

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \omega x \\ \omega y \end{pmatrix} \quad \omega^3 = 1$$

$$R^G = \mathbb{C}[x^3, x^2y, xy^2, y^3]$$

is the third Veronese subring of the polynomial ring. Here we get a ring of invariants that is not Gorenstein.
Example 1.4. $G = \langle \alpha, \beta \rangle \cong Q_8$ acts on $R = \mathbb{C}[x, y]$

\[
\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ix \\ -iy \end{pmatrix}, \quad \beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}
\]

$R^G = \mathbb{C}[xy(x^4 - y^4), x^4 + y^4, x^2 y^2]$

\[
\cong \mathbb{C}[u, v, w]/(u^2 + v^3 + w^3)
\]

The ring of invariants is the $D_4$ triple cone-point singularity seen in previous talks.

2. Ordinary Case

Let $|G| \in k^\times$ for this section.

Under this hypothesis, there are many strong theorems from which we can see when $R^G$ is "good".

**Theorem 2.1** (Shepard-Todd, Chevalley, Serre). $R^G$ is a polynomial ring $\iff$ $G$ is generated by pseudoreflections (elements $g$ that fix a hyperplane in the vector-space representation on $R_1$).

**Example 2.2.** $S_n$ is generated by transpositions. Any transposition certainly fixes every variable except for the two being switched, and fixes the sum of the two being switched. Thus, a transposition is a pseudoreflection, so [STCS] says the ring of invariants should be polynomial, as is the case.

**Example 2.3.** $A_n$ cannot be generated by transpositions; perhaps the best generating set consists of 3-cycles. These do not fix any hyperplanes. In fact, $A_n$ contains no pseudoreflections. The [STCS] theorem applies to conclude that $R^{A_n}$ is not polynomial.

The following theorem says that $R^G$ is always "good" in a homological sense.

**Theorem 2.4** (Hochster-Eagon). $R^G$ is Cohen-Macaulay

We briefly review a proof, since some of the concepts involved will show up later on in our discussion.

**Proof.** There is an $R^G$-linear map

\[
\text{Tr}^G : R \to R^G
\]

\[
r \mapsto \sum_{g \in G} g(r)
\]

Note that $r \in R^G \Rightarrow \text{Tr}^G(r) = |G|r$, so $\rho(r) = \frac{1}{|G|} \text{Tr}^G(r) = r$. Then, to the diagram

\[
\begin{array}{ccc}
R^G & \xrightarrow{1} & R^G \\
\downarrow i & & \downarrow \rho \\
R & \quad & \\
\end{array}
\]

apply the local cohomology functors with support in $R^G$. Note that $R^G$ expands to an $R_1$-primary ideal in $R$, and local cohomology is radically-invariant in the
support:

\[
\begin{array}{ccc}
H^i_{R_G}(R^G) & \xrightarrow{1_*} & H^i_{R_G}(R^G) \\
\downarrow{i_*} & & \downarrow{\rho_*} \\
H^i_{R_+}(R) & & 
\end{array}
\]

so that \(i_*\) is injective. But \(H^i_{R_+}(R) = 0\) for \(i < \dim R\) (itself is Cohen-Macaulay), so \(H^i_{R_G}(R^G) = 0\) for \(i < \dim R^G = \dim R\).

3. Modular Case

Let \(|G| \notin k^\times\) for this section, i.e., char \(k = p\) \(|G|\).

It is still true ([Serre]) that if \(R^G\) is a polynomial ring, then \(G\) is generated by pseudoreflections. The converse is false:

**Example 3.1** (Campbell-Hughes-Shank). \(G = \langle \alpha, \beta, \gamma \rangle \cong (k_+)^3\) acts on \(R = k[x_1, x_2, y_1, y_2]\)

\[
\begin{align*}
\alpha(x_i) &= \beta(x_i) = \gamma(x_i) = x_i, & i &= 1, 2 \\
\alpha(y_1) &= (y_1 + x_1, y_2), & \beta(y_1) &= (y_1, y_2 + x_2), & \gamma(y_1) &= (y_1 + x_1 + x_2, y_2 + x_1 + x_2) \\
R^G &= k[x_1, x_2, x_1 \prod_{g \in \langle \alpha \rangle} g(y_1) + x_2 \prod_{g \in \langle \beta \rangle} g(y_2), \prod_{g \in \langle \beta, \gamma \rangle} g(y_1), \prod_{g \in \langle \alpha, \gamma \rangle} g(y_2)] \\
&= k[x_1, x_2, w, z_1, z_2]/(w^p - x_1^p z_1 - x_2^p z_2 - w(\sum_{j=1}^{p} x_1^{p-j+1} x_2^j)^{p-1})
\end{align*}
\]

Note that \(\alpha, \beta, \gamma\) are all pseudoreflections.

What about the Cohen-Macaulay property?
This fails too:

**Example 3.2** (Bertin). \(G = \mathbb{Z}/4\mathbb{Z}\) acts on \(R = \mathbb{F}_2[x_1, x_2, x_3, x_4]\) by cyclically permuting variables. \(R^G\) is not Cohen-Macaulay.

This was the first example of a UFD that is not Cohen-Macaulay. Moreover, ([Fossum-Griffith]) \(\hat{R}^G\) is a complete UFD that is not Cohen-Macaulay.

In a certain sense, the Cohen-Macaulay property is somewhat restrictive in the modular case.

**Theorem 3.3** (Kemper). Let \(|G| = p^e\).
\(R^G\) is Cohen-Macaulay \(\implies G\) is generated by bireflections (elements \(g\) that fix a codimension 2 vector-subspace in the vector-space representation on \(R_1\))

**Remark 3.4.** There are examples of groups generated by pseudoreflections with non-Cohen Macaulay ring of invariants. Thus there are examples where [STCS] and [HE] fail simultaneously.

Determining when \(R^G\) is polynomial or Cohen-Macaulay are two driving questions in modular invariant theory. We consider a question related to both.
4. Splittings

Recall that the proof of [Hochster-Eagon] we considered shows that if the inclusion $R^G \hookrightarrow R$ is split, then $R^G$ is Cohen-Macaulay. Also, note that if $R^G$ is a polynomial ring, it is a Noether normalization for the Cohen-Macaulay ring $R$, so $R^G \hookrightarrow R$ is free, hence split. Put together,

$$R^G \text{ polynomial } \implies R^G \hookrightarrow R \text{ splits } \implies R^G \text{ Cohen-Macaulay}$$

When is $R^G \hookrightarrow R$ split?

**Remark 4.1.** $R^G \hookrightarrow R$ splits $\iff R^G$ $F$-regular, so this can be thought of as another variation on the question of when $R^G$ is “good”.

**Definition 4.2.** For a Cohen-Macaulay graded ring $S$, put

$$a(S) = \max\{d : [H^j_{S^n}(S)]_d \neq 0\}$$

**Lemma 4.3.** $a(R^G) \leq a(R)$

**Remark 4.4.** If $|G| \in k^\times$, $a(R^G) = a(R) \iff G \leq \text{SL}_k(R_1)$

**Theorem 4.5 (—).** Let $|G| \notin k^\times$. If $R^G \hookrightarrow R$ splits, then $a(R^G) < a(R)$.

**Corollary 4.6 (Singh, Smith, Glasbrenner).** If $p || A_n$, then $R^{A_n} \hookrightarrow R$ is not split, i.e. $R^{A_n}$ is not $F$-regular.

**Proof.** It is easy to see that $a(R^{A_n}) = -n$, noting that $\Delta^2 \in R^{S_n}$. $lacksquare$

**Corollary 4.7.** If $\Sigma \leq A_n$, $p || \Sigma$, then $R^\Sigma \hookrightarrow R$ is not split, i.e. $R^\Sigma$ is not $F$-regular.

We might also ask how different the notions considered above truly are.

**Conjecture 4.8.** Let $|G| = p^e$ with $p = \text{char } k$. If $R^G \hookrightarrow R$ splits then $R^G$ is actually a polynomial ring.