Additional Problems

1. In this problem, we will solve \( x^3 + x + 1 = 0 \). We’ll use a clever trick of Cardano.
   (a) Put \( x = s + t \) with \( t = \frac{-1}{3s} \) (so that \( 3st + 1 = 0 \)) and rewrite the given equation according to this substitution. The equation should simplify drastically. Why can we always find such \( s \) and \( t \)?
   (b) Find \( s^3 + t^3 \) and \( s^3t^3 \). Use these facts to find a quadratic equation which has \( s^3 \) and \( t^3 \) as its two roots.
   (c) Solve the quadratic equation above, and use this to solve for \( x \).

2. Let \( D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \) be an \( n \times n \) diagonal matrix (i.e., a matrix where the only non-zero entries are on the main diagonal).
   (a) Let \( A \) be any \( n \times p \) matrix. Prove that \((\text{row } i \text{ of } DA) = d_i(\text{row } i \text{ of } A)\).

   \textbf{Hint: Use what we know about the rows of a product of two matrices. Factor out a } d_i \text{ on one side and use the example from class.}

   (b) Let \( B \) be any \( m \times n \) matrix. Prove that \((\text{column } j \text{ of } BD) = d_j(\text{column } j \text{ of } B)\).

3. Read Example 2.4A in the text. What is the inverse of \( P \)? Compute \( P^{-1} \) times the \textbf{ones} vector and \( P^{-1} \) times the \textbf{powers} vector.

4. Here’s a direct proof (by using the multiplication formula) of the property \((AB)C = A(BC)\)
   Let \( A = [a_{ij}]_{i \times m} \), \( B = [b_{ij}]_{m \times n} \), \( C = [c_{ij}]_{n \times p} \). Then

   \[ (AB)C = \sum_{j=1}^{n} (AB)_{hj}c_{jk} = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{hi}b_{ij} \right) c_{jk} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{hi}b_{ij}c_{jk} \]

   \[ A(BC) = \sum_{i=1}^{m} a_{hi}(BC)_{ik} = \sum_{i=1}^{m} a_{hi} \left( \sum_{j=1}^{n} b_{ij}c_{jk} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{hi}b_{ij}c_{jk} \]

   Since each entry of \((AB)C\) and \(A(BC)\) agrees, the two matrices agree.
   Use the multiplication formula to show that the product of two lower triangular matrices is also lower triangular.

   \textbf{Hint: } \( L = [\ell_{ij}]_{n \times n} \) is lower triangular if \( \ell_{ij} = 0 \) for all \( i < j \).