2.5 Inverse Matrices

Suppose $A$ is a square matrix. We look for an "inverse matrix" $A^{-1}$ of the same size, such that $A^{-1} A$ equals $I$. Whatever $A$ does, $A^{-1}$ undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1} A x = x$. But $A^{-1}$ might not exist.

What a matrix mostly does is to multiply a vector $x$. Multiplying $A x = b$ by $A^{-1}$ gives $A^{-1} A x = A^{-1} b$. This is $x = A^{-1} b$. The product $A^{-1} A$ is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix $A^{-1}$ is called "$A$ inverse."

**DEFINITION** The matrix $A$ is invertible if there exists a matrix $A^{-1}$ such that

$$A^{-1} A = I \quad \text{and} \quad A A^{-1} = I.$$  

(1)

Not all matrices have inverses. This is the first question we ask about a square matrix: Is $A$ invertible? We don’t mean that we immediately calculate $A^{-1}$. In most problems we never compute it! Here are six "notes" about $A^{-1}$.

Note 1 The inverse exists if and only if elimination produces $n$ pivots (row exchanges are allowed). Elimination solves $A x = b$ without explicitly using the matrix $A^{-1}$.

Note 2 The matrix $A$ cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$(BA)C = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \quad (2)$$

This shows that a left-inverse $B$ (multiplying from the left) and a right-inverse $C$ (multiplying $A$ from the right to give $AC = I$) must be the same matrix.

Note 3 If $A$ is invertible, the one and only solution to $A x = b$ is $x = A^{-1} b$:

**Multiply** $A x = b$ by $A^{-1}$. Then $x = A^{-1} A x = A^{-1} b$.

Note 4 (Important) Suppose there is a nonzero vector $x$ such that $A x = 0$. Then $A$ cannot have an inverse. No matrix can bring $0$ back to $x$.

If $A$ is invertible, then $A x = 0$ can only have the zero solution $x = 0$. Then $A$ cannot have an inverse. No matrix can bring $0$ back to $x$.

Note 5 A $2 \times 2$ matrix is invertible if and only if $ad - bc$ is not zero:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (3)$$

This number $ad - bc$ is the determinant of $A$. A matrix is invertible if its determinant is not zero (Chapter 5). The test for $n$ pivots is usually decided before the determinant appears.
Chapter 2. Solving Linear Equations

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

If \( A = \begin{bmatrix} d_1 & \cdots & d_n \\ \vdots & \ddots & \vdots \\ d_1 & \cdots & d_n \end{bmatrix} \) then \( A^{-1} = \begin{bmatrix} 1/d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/d_n \end{bmatrix} \).

Example 1 The 2 by 2 matrix \( A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \) is not invertible. It fails the test in Note 5, because \( ad - bc = 2 - 2 = 0 \). It fails the test in Note 3, because \( Ax = 0 \) when \( x = (2, -1) \). It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix \( A \) into a zero row.

The Inverse of a Product \( AB \)

For two nonzero numbers \( a \) and \( b \), the sum \( a + b \) might or might not be invertible. The numbers \( a = 3 \) and \( b = -3 \) have inverses \( \frac{1}{3} \) and \(-1\). Their sum \( c = b = 0 \) has no inverse.

But the product \( ab = -9 \) does have an inverse, which is \( \frac{1}{9} \) times \(-1\).

For two matrices \( A \) and \( B \), the situation is similar. It is hard to say much about the invertibility of \( A + B \). But the product \( AB \) has an inverse, if and only if the two factors \( A \) and \( B \) are separately invertible (and the same size). The important point is that \( A^{-1} \) and \( B^{-1} \) come in reverse order.

If \( A \) and \( B \) are invertible then so is \( AB \). The inverse of a product \( AB \) is

\[
(AB)^{-1} = B^{-1}A^{-1}.
\]

(4)

To see why the order is reversed, multiply \( AB \) times \( B^{-1}A^{-1} \). Inside that is \( BB^{-1} = I \):

\[
\text{Inverse of } AB \quad (AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.
\]

We moved parentheses to multiply \( B^{-1}A^{-1} \) first. Similarly \( B^{-1}A^{-1} \) times \( AB \) equals \( I \). This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the socks. The same reverse order applies to three or more matrices:

\[
\text{Reverse order } \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.
\]

(5)

Example 2 Inverse of an elimination matrix. If \( E \) subtracts 5 times row 1 from row 2, then \( E^{-1} \) adds 5 times row 1 to row 2:

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Multiply \( EE^{-1} \) to get the identity matrix \( I \). Also multiply \( E^{-1}E \) to get \( I \). We are adding or subtracting the same 5 times row 1. Whether we add and then subtract (this is \( EE^{-1} \)) or subtract and then add (this is \( E^{-1}E \)), we are back at the start.

2.5. Inverse Matrices

For square matrices, an inverse on one side is automatically an inverse on the other side.

If \( AB = I \) then automatically \( BA = I \). In that case \( B \) is \( A^{-1} \). This is very useful to know but we are not ready to prove it.

Example 3 Suppose \( F \) subtracts 4 times row 2 from row 3, and \( F^{-1} \) adds it back:

\[
F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.
\]

Now multiply \( F \) by the matrix \( E \) in Example 2 to find \( FE \). Also multiply \( E^{-1} \) times \( F^{-1} \) to find \( (FE)^{-1} \). Notice the orders \( FE \) and \( E^{-1}F^{-1} \):

\[
FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.
\]

(6)

The result is beautiful and correct. The product \( FE \) contains "20" but its inverse doesn’t. \( E \) subtracts 5 times row 1 from row 2. Then \( F \) subtracts 4 times the new row 2 (changed by row 1) from row 3. In this order \( FE \), row 3 feels an effect from row 1.

In the order \( E^{-1}F^{-1} \), that effect does not happen. First \( F^{-1} \) adds 4 times row 2 to row 3. After that, \( E^{-1} \) adds 5 times row 1 to row 2. There is no 20, because row 3 doesn’t change again. In this order \( E^{-1}F^{-1} \), row 3 feels no effect from row 1.

In elimination order \( F \) follows \( E \). In reverse order \( E^{-1} \) follows \( F^{-1} \).

\( E^{-1}F^{-1} \) is quick. The multipliers 5, 4 fall into place below the diagonal of \( Y \).

This special multiplication \( E^{-1}F^{-1} \) and \( E^{-1}F^{-1}G^{-1} \) will be useful in the next section. We will explain it again, more completely. In this section our job is \( A^{-1} \), and we expect some serious work to compute it. Here is one way to organize that computation.

Calculating \( A^{-1} \) by Gauss-Jordan Elimination

I hinted that \( A^{-1} \) might not be explicitly needed. The equation \( Ax = b \) is solved by \( x = A^{-1}b \). But it is not necessary or efficient to compute \( A^{-1} \) and multiply it times \( b \).

Elimination goes directly to \( x \). Elimination is also the way to calculate \( A^{-1} \), as we now show. The Gauss-Jordan idea is to solve \( AA^{-1} = I \), finding each column of \( A^{-1} \).

\( A \) multiplies the first column of \( A^{-1} \) (call that \( x_1 \)) to give the first column of \( I \) (call that \( e_1 \)). This is our equation \( Ax_1 = e_1 = (1, 0, 0) \). There will be two more equations. Each of the columns \( x_1, x_2, x_3 \) of \( A^{-1} \) is multiplied by \( A \) to produce a column of \( I \):

\[
3 \text{ columns of } A^{-1} \quad A A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = [e_1 \ e_2 \ e_3] = I.
\]

(7)

To invert a 3 by 3 matrix \( A \), we have to solve three systems of equations: \( Ax_1 = e_1 \) and \( Ax_2 = e_2 = (0, 1, 0) \) and \( Ax_3 = e_3 = (0, 0, 1) \). Gauss-Jordan finds \( A^{-1} \) this way.
Chapter 2. Solving Linear Equations

2.5. Inverse Matrices

The elimination steps create the inverse matrix while changing A to I. For large matrices, we probably don't want A' at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular K' because it is an important example. We introduce the words symmetric, tridiagonal, and determinant:

1. K is symmetric across its main diagonal. So is K'.
2. K is tridiagonal (only three nonzero diagonals). But K' is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.
3. The product of pivots is \( \det(K) = 2 \). This number is the determinant of K.

The Gauss-Jordan method computes \( A^{-1} \) by solving all \( n \) equations together.

![Matrix Image](image)

The elimination steps create the inverse matrix while changing A to I. For large matrices, we probably don't want \( A' \) at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular \( K' \) because it is an important example. We introduce the words symmetric, tridiagonal, and determinant:

1. \( K \) is symmetric across its main diagonal. So is \( K' \).
2. \( K \) is tridiagonal (only three nonzero diagonals). But \( K' \) is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.
3. The product of pivots is \( 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 4 \). This number is the determinant of \( K \).

\[ K^{-1} \text{ involves division by the determinant} \quad K^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 4 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \] (8)

This is why an invertible matrix cannot have a zero determinant.

Example 4 Find \( A^{-1} \) by Gauss-Jordan elimination starting from \( A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 0 \\ 0 & 1 & -2 \end{bmatrix} \). There are two row operations and then a division to put 1's in the pivots:

\[ \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 & 1 \\ 4 & 7 & 0 & 1 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \]

That \( A^{-1} \) involves division by the determinant \( \det(A) = -4 \). The code for \( X = \text{inverse}(A) \) can use \texttt{rref}, the "row reduced echelon form" from Chapter 3:

\[ I = \text{eye}(n); \]

\[ R = \text{rref}([A \; I]); \]

\[ X = R(:,n+1:n); \]

\( A \) must be invertible, or elimination cannot reduce it to I (in the left half of \( R \)). Gauss-Jordan shows why \( A^{-1} \) is expensive. We must solve \( n \) equations for its \( n \) columns.

To solve \( A x = b \) without \( A^{-1} \), we deal with one column \( b \) to find one column \( x \).

In defense of \( A^{-1} \), we want to say that its cost is not \( n \) times the cost of one system \( Ax = b \). Surprisingly, the cost for \( n \) columns is only multiplied by 3. This saving is because the \( n \) equations \( A x_i = b_i \) all involve the same matrix \( A \). Working with the right sides is relatively cheap, because elimination only has to be done once on \( A \). The complete \( A^{-1} \) needs \( n^3 \) elimination steps, where a single \( x \) needs \( n^3/3 \). The next section calculates these costs.
We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: \( A^{-1} \text{ exists exactly when } A \text{ has a full set of } n \text{ pivots.} \)

(Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With a pivot, elimination solves all the equations \( Ax_i = e_i. \) The columns \( x_i \) go into \( A^{-1}, \) Then \( AA^{-1} = I \) and \( A^{-1} \) is at least a right-inverse.

2. Elimination is really a sequence of multiplications by \( E' \)'s and \( P' \)'s and \( D^{-1}; \)

\[
(D^{-1} \ldots E \ldots P \ldots E)A = I. \tag{9}
\]

Left-inverse

\( D^{-1} \) divides by the pivots. The matrices \( E \) produce zeros below and above the pivots. \( P \) will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a left-inverse. With a pivot we have reached \( A^{-1}A = I. \)

The right-inverse equals the left-inverse. That was Note 2 as the start of this section.

So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will show that \( A \) must have \( n \) pivots if \( AC = I. \) (Then we deduce that \( C \) is also a left-inverse and \( CA = I \).) Here is one route to those conclusions:

1. If \( A \) doesn't have \( n \) pivots, elimination will lead to a zero row.

2. Those elimination steps are taken by an invertible \( M. \) So a row of \( MA \) is zero.

3. If \( AC = I \) had been possible, then \( MAC = M. \) The zero row of \( MA, \) times \( C, \) gives a zero row of \( M \) itself.

4. An invertible matrix \( M \) can't have a zero row! \( A \) must have \( n \) pivots if \( AC = I. \)

That argument took four steps, but the outcome is short and important.

Elimination gives a complete test for invertibility of a square matrix. \( A^{-1} \text{ exists (and Gauss-Jordan finds it) exactly when } A \text{ has a pivot.} \) The argument above shows more:

\[
\text{If } AC = I \text{ then } CA = I \text{ and } C = A^{-1}.
\]

Example 5 If \( L \) is lower triangular with 1's on the diagonal, so is \( L^{-1}. \)

A triangular matrix is invertible if and only if no diagonal entries are zero.

\[ L \text{ has 1's so } L^{-1} \text{ also has 1's. Use the Gauss-Jordan method to construct } L^{-1}. \text{ Start by subtracting multiples of pivot rows from rows below. Normally this gets us halfway to the inverse, but for } L \text{ it gets us all the way. } L^{-1} \text{ appears on the right when } I \text{ appears on the left. Notice how } L^{-1} \text{ contains } 11, \text{ from 3 times 5 minus 4.} \]

2.5. Inverse Matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 & 1 \\
4 & 5 & 1 & 0 & 0
\end{bmatrix} = \begin{bmatrix} L & I \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 & 1 \\
4 & 5 & 1 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1
\end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix} = \begin{bmatrix} I & \text{sum matrix} \end{bmatrix}.
\]

If I change \( a_{13} = 3, \) then all rows of \( A \) add to zero. The equation \( Ax = 0 \) will now have the nonzero solution \( x = (1, 1, 1). \) A clear signal: This new \( A \) can't be inverted.
2.5 B Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to \( Ax = 0 \)) for the other three. The matrices are in the order \( A, B, C, D, E \):

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

Solution

\[
B^{-1} = \begin{bmatrix}
8 & -3 \\
-8 & 4
\end{bmatrix}, \quad C^{-1} = \begin{bmatrix}
0 & 6 \\
6 & -6
\end{bmatrix}, \quad S^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

\( A \) is not invertible because its determinant is \( 4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0 \). \( D \) is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. \( E \) is not invertible because a combination of the columns (the second column is subtracted, \( E \) is equal to 0) has the solution \( x = (-1, 1, 0) \).

Of course all three reasons for noninvertibility would apply to each of \( A, D, E \).

2.5 C Apply the Gauss-Jordan method to invert this triangular "Pascal matrix" \( L \). You see Pascal's triangle—adding each entry to the entry on its left gives the entry below. The entries of \( L \) are "binomial coefficients". The next row would be 1, 4, 6, 4, 1.

\[
\text{Triangular Pascal matrix } L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix} = \text{abc(pascal}(4,1))
\]

Solution

Gauss-Jordan starts with \([ L ] I\) and produces zeros by subtracting row 1:

\[
[L] I = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & -1 & 1 & 0 \\
0 & 3 & 3 & 1 & -1 & 0 & 1
\end{bmatrix}
\]

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = [ L^{-1} ].
\]

All the pivots were 1! So we didn't need to divide rows by pivots to get 1. The inverse matrix \( L^{-1} \) looks like \( L \) itself, except odd-numbered diagonals have minus signs.

The same pattern continues to \( n \) by \( n \) Pascal matrices, \( L^{-1} \) has "alternating diagonals".

2.5. Inverse Matrices

Problem Set 2.5

1 Find the inverses (directly or from the 2 by 2 formula) of \( A, B, C \):

\[
A = \begin{bmatrix}
3 & 1 \\
4 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 0 \\
4 & 2
\end{bmatrix}, \quad C = \begin{bmatrix}
3 & 4 \\
5 & 7
\end{bmatrix}
\]

2. For these "permutation matrices" find \( P^{-1} \) by trial and error (with 1's and 0's):

\[
P = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

3. Solve for the first column \((x, y)\) and second column \((t, z)\) of \( A^{-1} \):

\[
\begin{bmatrix}
10 & 20 \\
20 & 50
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad \text{and } \begin{bmatrix}
10 & 20 \\
20 & 50
\end{bmatrix} \begin{bmatrix}
t \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

4. Show that \( \frac{1}{4} I \) is not invertible by trying to solve \( A A^{-1} = I \) for column 1 of \( A^{-1} \):

\[\begin{bmatrix}
1 & 2 \\
3 & 6
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
6 \\
0
\end{bmatrix}\]

For a different \( A \), could column 1 of \( A^{-1} \) be possible to find but not column 2?

5. Find an upper triangular \( U \) (not diagonal) with \( U^2 = I \) which gives \( U = U^{-1} \).

(a) If \( A \) is invertible and \( AB = AC \), prove quickly that \( B = C \).

(b) If \( A = \begin{bmatrix} 1 & 1 \end{bmatrix} \), find two different matrices such that \( AB = AC \).

(Important) If \( A \) has row 1 + row 2 = row 3, show that \( A \) is not invertible:

(a) Explain why \( Ax = (1, 0, 0) \) cannot have a solution.

(b) Which right sides \((b_1, b_2, b_3)\) might allow a solution to \( Ax = b \)?

(c) What happens to row 3 in elimination?

8. If \( A \) has column 1 + column 2 = column 3, show that \( A \) is not invertible:

(a) Find a nonzero solution \( x \) to \( Ax = 0 \). The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

9. Suppose \( A \) is invertible and you exchange its first two rows to reach \( B \). Is the new matrix \( B \) invertible and how would you find \( B^{-1} \) from \( A^{-1} \)?

10. Find the inverses (in any legal way) of

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
3 & 2 & 0 \\
0 & 3 & 0 \\
5 & 0 & 0
\end{bmatrix}
\]
Chapter 2. Solving Linear Equations

11 (a) Find invertible matrices $A$ and $B$ such that $A + B$ is not invertible.
(b) Find singular matrices $A$ and $B$ such that $A + B$ is invertible.

12 If the product $C = AB$ is invertible (and $A$ and $B$ are square), then $A$ itself is invertible.

13 (a) Find a formula for $A^{-1}$ that involves $C^{-1}$ and $B$.
(b) Find a formula for $B^{-1}$ that involves $M^{-1}$ and $A$ and $C$.

14 If you add row 1 of $A$ to row 2 to get $B$, how do you find $B^{-1}$ from $A^{-1}$?

Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A$ is ______.

15 Prove that a matrix with a column of zeros cannot have an inverse.

16 Multiply $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ times $\begin{bmatrix} 4 \\ -b \\ a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

17 (a) What matrix $E$ has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
(b) What single matrix $L$ has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18 If $B$ is the inverse of $A^2$, show that $AB$ is the inverse of $A$.

19 Find the numbers $a$ and $b$ that give the inverse of $5 \cdot \text{eye}(4) - \text{ones}(4,4)$:

\[
\begin{bmatrix}
4 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}^{-1} =
\begin{bmatrix}
a & b & b & b \\
 b & a & b & b \\
b & b & a & b \\
b & b & b & a
\end{bmatrix}.
\]

What are $a$ and $b$ in the inverse of $6 \cdot \text{eye}(5) - \text{ones}(5,5)$?

20 Show that $A = 4 \cdot \text{eye}(4) - \text{ones}(4,4)$ is not invertible; Multiply $A \cdot \text{ones}(4,1)$.

21 There are sixteen $2 \times 2$ matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating $A^{-1}$.

22 Change $I$ into $A^{-1}$ as you reduce $A$ to $I$ (by row operations):

\[
[A I] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

and

\[
[A I] = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 9 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

23 Follow the 3 by 3 text example but with plus signs in $A$. Eliminate above and below the pivots to reduce $[A I]$ to $[I A^{-1}]$:

\[
[A I] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

24 Use Gauss-Jordan elimination on $[U I]$ to find the upper triangular $U^{-1}$:

\[
UU^{-1} = I
\]

25 Find $A^{-1}$ and $B^{-1}$ (if they exist) by elimination on $[A I]$ and $[B I]$:

\[
A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}
\]

26 What three matrices $E_{12}$ and $E_{13}$ and $D_{-1}$ reduce $A = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ to the identity matrix? Multiply $D_{-1}E_{12}E_{13}$ to find $A^{-1}$.

27 Invert these matrices $A$ by the Gauss-Jordan method starting with $[A I]$:

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}
\]

and

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}
\]

28 Exchange rows and continue with Gauss-Jordan to find $A^{-1}$:

\[
[A I] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}
\]

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's down the main diagonal is invertible.
(c) If $A$ is invertible then $A^{-1}$ and $A^2$ are invertible.

30 For which three numbers $c$ is this matrix not invertible, and why not?

\[
A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}
\]

31 Prove that $A$ is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or $A^{-1}$):

\[
A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}
\]
Chapter 2. Solving Linear Equations

32 This matrix has a remarkable inverse. Find $A^{-1}$ by elimination on $[A | I]$. Extend to a $5 \times 5$ "alternating matrix" and guess its inverse; then multiply to confirm.

Invert $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and solve $Ax = (1, 1, 1, 1, 1)$.

33 Suppose the matrices $P$ and $Q$ have the same rows as $I$ but in any order. They are "permutation matrices". Show that $P - Q$ is singular by solving $(P - Q)x = 0$.

34 Find and check the inverses (assuming they exist) of these block matrices:

$$
\begin{bmatrix}
I & 0 \\
C & I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & D
\end{bmatrix}
$$

35 Could a $4 \times 4$ matrix $A$ be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of $B$ contains $0, 1, 2, 3$ in some order?

36 In the Worked Example 2.5 C, the triangular Pascal matrix $L$ has an inverse with "alternating diagonals". Check that this $L^{-1}$ is $DLD$, where the diagonal matrix $D$ has alternating entries $1, -1, 1, -1$. Then $LDL = I$, so what is the inverse of $LD$? (Is it $pascal(4, 1)$?)

37 The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse invltb(6). Then ask it to compute invltb(6). How can these be different, when the computer never makes mistakes?

(a) Use inv(P) to invert MATLAB's $4 \times 4$ symmetric matrix $P = pascal(4)$.
(b) Create Pascal's lower triangular $L = abs(pascal(4))$ and test $P = LL'$.

39 If $A = ones(4)$ and $b = rand(4, 1)$, how does MATLAB tell you that $Ax = b$ has no solution? For the special $b = ones(4, 1)$, which solution to $Ax = b$ is found by $A\backslash b$?

Challenge Problems

40 (Recommended) $A$ is a $4 \times 4$ matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find $A^{-1}$ for this bidiagonal matrix.

41 Suppose $E_1, E_2, E_3$ are $4 \times 4$ identity matrices, except $E_1$ has $a, b, c$ in column 1 and $E_2$ has $d, e$ in column 2 and $E_3$ has $f$ in column 3 (below the 1's). Multiply $L = E_1 E_2 E_3$ to show that all these nonzeros are copied into $L$.

$E_1 E_2 E_3$ is in the opposite order from elimination (because $E_3$ is acting first). But $E_1 E_2 E_3 = L$ is in the correct order to invert elimination and recover $A$.

2.5. Inverse Matrices

42 Direct multiplications 1–4 give $MM^{-1} = I$, and I would recommend doing #3. $M^{-1}$ shows the change in $A^{-1}$ (useful to know) when a matrix is subtracted from $A$:

- $M = I - uv$ and $M^{-1} = I + uv/(1 - uu')$ (rank 1 change in $I$)
- $M = I - uv$ and $M^{-1} = A^{-1} + A^{-1}uvA^{-1}/(1 - A^{-1}v')$
- $M = I - uv$ and $M^{-1} = I_n + U(L_n - V_n)V_n'$
- $M = A - U^{-1}V'W^{-1}V'$ and $M^{-1} = A^{-1} + A^{-1}U(W - V A^{-1}U)^{-1}V'W^{-1}V$ $

The Woodbury-Morrison formula 4 is the "matrix inversion lemma" in engineering. The Kalman filter for solving block tridiagonal systems uses formula 4 at each step. The four matrices $M^{-1}$ are in diagonal blocks when inverting these block matrices ($r$ is 1 by $n$, $u$ is $n$ by 1, $V$ is $m$ by $n$, $U$ is $n$ by $m$).

$$
\begin{bmatrix}
I & 0 \\
I & U
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
V & L
\end{bmatrix}
\begin{bmatrix}
I & U \\
V & W
\end{bmatrix}
$$

43 Second difference matrices have beautiful inverses if they start with $T_{11} = 1$ (instead of $K_{11} = 2$). Here is the $3 \times 3$ triangular matrix $T$ and its inverse:

$$
T_{11} = T = \begin{bmatrix}
1 & -1 & 0 \\
0 & -1 & -1 \\
0 & -1 & 0
\end{bmatrix}
\quad T^{-1} = \begin{bmatrix}
3 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

One approach is Gauss-Jordan elimination on $[T I]$. That seems too mechanical. I would rather write $T$ as the product of first differences $L$ times $U$. The inverses of $L$ and $U$ in Worked Example 2.5 A are sum matrices, so here are $T$ and $T^{-1}$:

$$
LU = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\quad U^{-1}L^{-1} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

44 Here are two more difference matrices, both important. But are they invertible?

Cyclic $C = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}$

Free ends $F = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}$

One test is elimination—the fourth pivot fails. Another test is the determinant, we don't want that. The best way is much faster, and independent of matrix size:

Produce $x \neq 0$ so that $C x = 0$. Do the same for $F x = 0$. Not invertible.

Show how both equations $Cx = b$ and $Fx = b$ lead to $0 = b_1 + b_2 + \cdots + b_n$. There is no solution for other $b$. 

45 Elimination for a 2 by 2 block matrix: When you multiply the first block row by $CA^{-1}$ and subtract from the second row, the “Schur complement” $S$ appears:

$$
\begin{bmatrix}
I & 0 \\
-C^{-1}I & I
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
A & B \\
0 & S
\end{bmatrix}
$$

A and $D$ are square

$S = D - CA^{-1}B$.

Multiply on the right to subtract $A^{-1}B$ times block column 1 from block column 2.

$$
\begin{bmatrix}
A & B \\
0 & S
\end{bmatrix}
\begin{bmatrix}
I & -A^{-1}B \\
0 & I
\end{bmatrix}
= ?
$$

Find $S$ for

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
2 & 3 & 1 \\
4 & 1 & 0 \\
4 & 0 & 1
\end{bmatrix}
$$

The block pivots are $A$ and $S$. If they are invertible, so is $\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$.

46 How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.

2.6 Elimination = Factorization: $A = LU$

Students often say that mathematics courses are too theoretical. Well, not this section. It is almost purely practical. The goal is to describe Gaussian elimination in the most useful way. Many key ideas of linear algebra, when you look at them closely, are really factorizations of a matrix. The original matrix $A$ becomes the product of two or three special matrices. The first factorization—also the most important in practice—comes now from elimination. The factors $L$ and $U$ are triangular matrices. The factorization that comes from elimination is $A = LU$.

We already know $U$, the upper triangular matrix with the pivots on its diagonal. The elimination steps take $A$ to $U$. We will show how reversing those steps (taking $U$ back to $A$) is achieved by a lower triangular $L$. The entries of $L$ are exactly the multipliers $\ell_{ij}$—which multiplied the pivot row $j$ when it was subtracted from row $i$.

Start with a 2 by 2 example. The matrix $A$ contains 2, 1, 6, 8. The number to eliminate is 6. Subtract 3 times row 1 from row 2. That step is $E_{21}$ in the forward direction with multiplier $\ell = 3$. The return step from $U$ to $A$ is $L = E_1^{-1}$ (an addition using $+3$):

$$
\begin{align*}
\text{Forward from } A & \text{ to } U: & E_{21}A &= \begin{bmatrix}
1 & 0 \\
-3 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
6 & 8
\end{bmatrix}
= \begin{bmatrix}
2 & 1 \\
0 & 5
\end{bmatrix}
= U \\
\text{Back from } U & \text{ to } A: & E_{21}^{-1}U &= \begin{bmatrix}
1 & 0 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
6 & 8
\end{bmatrix}
= \begin{bmatrix}
6 & 8 \\
0 & 5
\end{bmatrix}
= A.
\end{align*}
$$

The second line is our factorization $LU = A$. Instead of $E_{21}^{-1}$ we write $L$. Move now to larger matrices with many $E$’s. Then $L$ will include all their inverses.

Each step from $A$ to $U$ multiplies by a matrix $E_{ij}$ to produce zero in the $(i, j)$ position. To keep this clear, we stay with the most frequent case—when no row exchanges are involved. If $A$ is 3 by 3, we multiply by $E_{21}$ and $E_{31}$ and $E_{32}$. The multipliers $\ell_{ij}$ produce zeros in the $(2, 1)$ and $(3, 1)$ and $(3, 2)$ positions—all below the diagonal. Elimination ends with the upper triangular $U$.

Now move those $E$’s onto the other side, where their inverses multiply $U$:

$$(E_{32}E_{31}E_{21})A = U \quad \text{becomes} \quad A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U \quad \text{which is} \quad A = LU. \quad (1)$$

The inverses go in opposite order, as they must. That product of three inverses is $L$. We have reached $A = LU$. Now we stop to understand it.

Exploration and Examples

First point: Every inverse matrix $E^{-1}$ is lower triangular. Its off-diagonal entry is $\ell_{ij}$, to undo the subtraction produced by $-\ell_{ij}$. The main diagonals of $E$ and $E^{-1}$ contain 1’s. Our examples above had $E_{21} = 3$ and $E = \begin{bmatrix}
1 & 1 \\
-3 & 1
\end{bmatrix}$ and $L = E^{-1} = \begin{bmatrix}
1 & 1 \\
3 & 1
\end{bmatrix}$.

Second point: Equation (1) shows a lower triangular matrix (the product of the $E_{ij}$) multiplying $A$. It also shows all the $E_{ij}^{-1}$ multiplying $U$ to bring back $A$. This lower triangular product of inverses is $L$. 

Chapter 2. Solving Linear Equations
2.6. Elimination = Factorization: \( A = LU \)

Row of \( U \), we subtract multiples of earlier rows of \( U \) (not rows of \( A \)):

\[
\text{Row 3 of } U = \text{Row 3 of } A - \xi_3 \text{(Row 1 of } U) - \xi_3 \text{(Row 2 of } U).
\]

Rewrite this equation to see that the row \( [\xi_3 \quad \xi_3 \quad 1] \) is multiplying \( U \):

\[
\text{Row 3 of } U = \xi_3 \text{(Row 1 of } U) + \xi_3 \text{(Row 2 of } U) + 1 \text{(Row 3 of } U).
\]

This is exactly row 3 of \( A = LU \). That row of \( U \) holds \( \xi_3, \xi_3, 1 \). All rows look like this, whatever the size of \( U \). With no row exchanges, we have \( A = LU \).

**Better balance.** The \( LU \) factorization is "unsymmetric" because \( U \) has the pivots on its diagonal where \( L \) has 1's. This is easy to change. Divide \( U \) by a diagonal matrix \( D \) that contains the pivots. That leaves a new matrix with 1's on the diagonal:

\[
\text{Split } U \text{ into } D U = \begin{bmatrix} d_1 & d_2 & \cdots & d_n \\ \end{bmatrix} \begin{bmatrix} u_{11}/d_1 & u_{13}/d_3 & \cdots & u_{1n}/d_n \\ 1 & u_{23}/d_3 & \cdots & u_{2n}/d_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.
\]

It is convenient (but a little confusing) to keep the same letter \( U \) for this new upper triangular matrix. It has 1's on the diagonal (like \( L \)). Inversely, normal \( LU \), the new form thus \( D \) in the middle: **Lower triangular \( L \) times diagonal \( D \) times upper triangular \( U \)**.

**The triangular factorization can be written** \( A = LU \) or \( A = LDL^T \).

Wherever you see \( LDL \), it is understood that \( U \) has 1's on the diagonal. Each row is divided by its first nonzero entry—the pivot. Then \( L \) and \( U \) are treated exactly in \( LDL \):

\[
\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 3 & 5 \\ \end{bmatrix} \text{ splits further into } \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \end{bmatrix}.
\]

The pivots 2 and 5 went into \( D \). Dividing the rows by 2 and 5 left the rows \( [1 \quad 0] \) and \( [1 \quad 0] \) in the new \( U \) with diagonal ones. The multiplier 3 is still in \( L \).

My own lectures sometimes stop at this point. The next paragraphs show how elimination codes are organized, and how long they take. If MATLAB (or any software) is available, you can measure the computing time by just counting the seconds.

**One Square System = Two Triangular Systems**

The matrix \( L \) contains our memory of Gaussian elimination. It holds the numbers that multiplied the pivot rows, before subtracting them from lower rows. When do we need this record and how do we use it in solving \( Ax = b \)?

We need \( L \) as soon as there is a right side \( b \). The factors \( L \) and \( U \) were completely decided by the left side (the matrix \( A \)). On the right side of \( Ax = b \), we use \( L^{-1} \) and then \( U^{-1} \). That **Solve** step deals with two triangular matrices.
2.6. Elimination = Factorization: $A = LU$

The first stage of elimination, on column 1, produces zeros below the first pivot. To find each new entry below the pivot row requires one multiplication and one subtraction. We will count this first stage as $n^2$ multiplications and $n^2$ subtractions. It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the second column below the second pivot. The working matrix is now of size $n-1$. Estimate this stage by $(n-1)^2$ multiplications and subtractions.

The matrices are getting smaller as elimination goes forward. The rough count to reach $U$ is the sum of squares $n^2 + (n-1)^2 + \ldots + 2^2 + 1^2$.

There is an exact formula $\frac{1}{2}n(n + \frac{1}{2})(n + 1)$ for this sum of squares. When $n$ is large, the $\frac{1}{2}$ and the $\frac{1}{2}$ are not important. The number of multiplies is $\frac{1}{2}n^3$. The sum of squares is like the integral of $x^3$! The integral from 0 to $n$ is $\frac{1}{4}n^4$.

**Elimination on A requires about $\frac{1}{2}n^3$ multiplications and $\frac{1}{2}n^3$ subtractions.**

What about the right side $b$? Going forward, we subtract multiples of $b_1$ from the lower components $b_2, \ldots, b_n$. This is $n-1$ steps. The second stage takes only $n-2$ steps, because $b_1$ is not involved. The last stage of forward elimination takes one step.

Now start back substitution. Computing $x_n$ uses one step (divide by the last pivot). The next unknown uses two steps. When we reach $x_1$ it will require $n$ steps ($n-1$ substitutions of the other unknowns, then division by the first pivot). The total count on the right side, from $b$ to $c$ to $x$—forward to the bottom and back up to the top—is exactly $n^2$:

$$(n-1) + (n-2) + \ldots + 1 + [1 + 2 + \ldots + (n-1) + n] = n^2.$$  \hfill (6)

To see that sum, pair off $(n-1)$ with 1 and $(n-2)$ with 2. The pairings leave $n$ terms, each equal to $n$. That makes $n^2$. The right side costs a lot less than the left side!

**Solve Each right side needs $n^2$ multiplications and $n^2$ subtractions.**

A *band matrix* $b$ has only $w$ nonzero diagonals below and also above its main diagonal. The zero entries outside the band stay zero in elimination (zeros in $L$ and $U$). Clearing out the first column needs $n^2$ multiplications and subtractions ($w$ zeros to be produced below the pivot, each one using a pivot row of length $w$). Then clearing out all $n$ columns, to reach $U$, needs no more than $n w^2$. This saves a lot of time.

**Band matrices**

<table>
<thead>
<tr>
<th>Factor</th>
<th>change $\frac{1}{3} n^3$ to $n w^2$</th>
<th>Solve</th>
<th>change $n^2$ to $2 n w$</th>
</tr>
</thead>
</table>

Here are codes to factor $A$ into $LU$ and to solve $Ax = b$. The Teaching Cose slu stops right away if a number smaller than the tolerance "tol" appears in a pivot position. The Teaching Codes are on web.mit.edu/18.086/WWW. Professional codes will look down each column for the largest available pivot, to exchange rows and continue solving.

MATLAB’s backslash command $x = A \backslash b$ combines Factor and Solve to reach $x$. 

---

2. Factor into $L$ and $U$, by elimination on the left side matrix $A$.

3. Solve (forward elimination on $b$ using $L$, then back substitution for $x$ using $U$).

Earlier, we worked on $A$ and $b$ at the same time. No problem with that—just aumment to $\{A, b\}$. But most computer codes keep the two sides separate. The memory of eliminating $b$ held in $L$ and $U$, to process $b$ whenever we want to. The User’s Guide to elimination is held in $L$ and $U$, to process $b$ whenever we want to. The User’s Guide to elimination is held in $L$ and $U$.
2.6. Elimination = Factorization: $A = LU$

**REVIEW OF THE KEY IDEAS**

1. Gaussian elimination (with no row exchanges) factors $A$ into $L$ times $U$.
2. The lower triangular $L$ contains the numbers $\ell_{ij}$ that multiply pivot rows, going from $A$ to $U$. The product $LU$ adds those rows back to recover $A$.
3. On the right side we solve $Lc = b$ (forward) and $Ux = c$ (backward).
4. Factor: There are $\frac{1}{2}(n^2 - n)$ multiplications and subtractions on the left side.
5. Solve: There are $n^2$ multiplications and subtractions on the right side.
6. For a band matrix, change $\frac{1}{2}n^2$ to $ro$ and change $n^2$ to $2ro$.

**WORKED EXAMPLES**

2.6 A The lower triangular Pascal matrix $L$ contains the famous "Pascal triangle". Gauss-Jordan found its inverse in the worked example 2.5 C. This problem connects $L$ to the symmetric Pascal matrix $P$ and the upper triangular $U$. The symmetric $P$ has Pascal’s triangle tilted, so each entry is the sum of the entry above and the entry to the left.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$LU$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1 1 1]</td>
<td>[1 1 1]</td>
</tr>
<tr>
<td>[1 2 3]</td>
<td>[1 0 0]</td>
</tr>
<tr>
<td>[1 3 6 10]</td>
<td>[0 1 2 3]</td>
</tr>
<tr>
<td>[1 4 10 20]</td>
<td>[0 0 1 3]</td>
</tr>
</tbody>
</table>

Then predict and check the next row and column for 3 by 5 Pascal matrices.

**Solution** You could multiply $LU$ to get $P$. Better to start with the symmetric $P$ and reach the upper triangular $U$ by elimination:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 3 & 6 & 10 & 0 \\ 1 & 4 & 10 & 20 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 3 & 6 & 10 & 0 \\ 1 & 4 & 10 & 20 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} = U.$$
Chapter 2. Solving Linear Equations

You might expect the MATLAB command `lu(pascal(n))` to produce these $L$ and $U$. That doesn't happen because the `lu` subroutine chooses the largest available pivot in each column. The second pivot will change from 1 to 3. But a "Cholesky factorization" does to row exchanges: $U = cholesky(pascal(4))$

The full proof of $P = LU$ for all Pascal sizes is quite fascinating. The paper "Pascal Matrices" is on the course web page web.mit.edu/18.06 which is also available through MIT's OpenCourseWare at ocw.mit.edu. These Pascal matrices have many remarkable properties—we will see them again.

2.6B The problem is: Solve $Px = b = (1, 0, 0, 0)$. This right side is column of $I$ means that $x$ will be the first column of $P^{-1}$. That is Gauss-Jordan, matching the columns of $P^{-1} = I$. We already know the Pascal matrices $L$ and $U$ as factors of $P$.

Two triangular systems $Le = b$ (forward) $Ux = c$ (back).

Solution. The lower triangular system $Le = b$ is solved to form:

$c_1 + c_2 = 1$ gives $c_2 = -1$
$c_1 + 2e_2 + c_3 = 0$ gives $c_3 = 5$
$c_1 + 3c_2 + 3c_3 + c_4 = 10$ gives $c_4 = 10$

Forward elimination is multiplication by $L^{-1}$. It produces the upper triangular system:

\[
\begin{align*}
E_1 & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix} \\
E_2 & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix} \\
E_3 & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix} \\
E_4 & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix} \\
\end{align*}
\]

Then $E_1E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix}$ produces the upper triangular sytem $E_1E_4x = c$ comes from the original $x + y + z = 11$ in $Ax = b$ by subtracting $\ell_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ times equation 1 and $\ell_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ times the final equation 2. Reverse that to recover $[1; 3; 0; 11]$ in the last row of $A$ and $b$ from the final $[1; 1; 1; 5]$ and $[0; 1; 2; 0]$ in $e$ and $c$.

Row 3 of $[A \ b] = (\ell_3$ Row 1 + $\ell_2$ Row 2 + 1 Row 3) of $[U \ c]$. In matrix notation this is multiplication by $L$. So $A = LU$ and $b = Le$.

What are the 3 by 3 triangular systems $Le = b$ and $Ux = c$ from Problem 3? Check that $e = (5, 2, 2)$ solves the first one. Which $x$ solves the second one?

What matrix $E$ puts $A$ into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor $A$ into $LU$:

\[
A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \\ \end{bmatrix}
\]

6 What two elimination matrices $E_{21}$ and $E_{32}$ put $A$ into upper triangular form $E_{32}E_{21}A = U$? Multiply by $E_{32}E_{21}$ to factor $A$ into $LU$:

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \\ \end{bmatrix}
\]

7 What three elimination matrices $E_{21}, E_{31}, E_{42}$ put $A$ into its upper triangular form $E_{42}E_{31}E_{21}A = U$? Multiply by $E_{42}E_{31}E_{21}$ to factor $A$ into $LU$ times $U$:

\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \\ \end{bmatrix} L = E_{42}^{-1}E_{31}^{-1}E_{21}^{-1}
\]

8 Suppose $A$ is already lower triangular with 1's on the diagonal. Then $U = I$.

\[
A = L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \end{bmatrix}
\]

The elimination matrices $E_{21}, E_{31}, E_{42}$ contain $a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{21}E_{31}$ to find the single matrix $E$ that produces $EA = I$.

(b) Multiply $E_{21}E_{31}E_{42}$ to bring back $L$ (nicer than $E$).
9. When zero appears in a pivot position, \( A = LU \) is not possible! (We are requiring nonzero pivots in \( U \).) Show directly why these are both impossible:

\[
\begin{bmatrix}
0 & 1 \\
2 & 3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
\ell & 1
\end{bmatrix}
\begin{bmatrix}
d & e & g \\
\ell & 1 & f \\
m & n & l
\end{bmatrix}
\]

This difficulty is fixed by a row exchange. That needs a "permutation" \( P \).

10. Which number \( c \) leads to zero in the second pivot position? A row exchange is needed and \( A = LU \) will not be possible. Which \( c \) produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

\[
A = \begin{bmatrix}
1 & c & 0 \\
2 & 4 & 1 \\
3 & 5 & 1
\end{bmatrix}
\]

11. What are \( L \) and \( D \) (the diagonal pivot matrix) for this matrix \( A \)? What is \( U \) in \( A = LU \) and what is the new \( U \) in \( A = LDU \)?

Already triangular:

\[
A = \begin{bmatrix}
2 & 4 & 8 \\
0 & 3 & 9 \\
0 & 0 & 7
\end{bmatrix}
\]

12. \( A \) and \( B \) are symmetric across the diagonal (because \( 4 = 4 \)). Find their triple-factorizations \( LDU \) and say how \( U \) is related to \( L \) for these symmetric matrices:

Symmetric:

\[
A = \begin{bmatrix}
2 & 4 \\
4 & 11
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & 4 & 0 \\
4 & 12 & 4 \\
0 & 4 & 0
\end{bmatrix}
\]

13. (Recommended) Compute \( L \) and \( U \) for the symmetric matrix \( A \):

\[
A = \begin{bmatrix}
a & a & a \\
ab & b & b \\
ab & ab & c \\
ab & ab & c
\end{bmatrix}
\]

Find four conditions on \( a, b, c, d \) to get \( A = LU \) with four pivots.

14. This nonsymmetric matrix will have the same \( L \) as in Problem 13.

\[
A = \begin{bmatrix}
ar & r & r \\
abr & s & s \\
abr & ab & c \\
abr & ab & c
\end{bmatrix}
\]

Find the four conditions on \( a, b, c, d, r, s, t \) to get \( A = LU \) with four pivots.

2.6. Elimination = Factorization: \( A = LU \)

Problems 15-16 use \( L \) and \( U \) (without needing \( A \)) to solve \( Ax = b \).

15. Solve the triangular system \( Lc = b \) to find \( c \). Then solve \( Ux = c \) to find \( x \):

\[
L = \begin{bmatrix}
1 & 0 \\
4 & 1
\end{bmatrix}
\quad \text{and} \quad
U = \begin{bmatrix}
2 & 4 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
2 \\
11
\end{bmatrix}
\]

For safety multiply \( LU \) and solve \( Ax = b \) as usual. Circle \( c \) when you see it.

16. Solve \( Lc = b \) to find \( c \). Then solve \( Ux = c \) to find \( x \). What was \( A \)?

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad
U = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
\]

17. (a) When you apply the usual elimination steps to \( L \), what matrix do you reach?

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
\ell_{11} & 1 & 0 \\
\ell_{11} & \ell_{32} & 1
\end{bmatrix}
\]

(b) When you apply the same steps to \( I \), what matrix do you get?

(c) When you apply the same steps to \( 3 \), what matrix do you get?

18. If \( A = LDU \) and also \( A = L_1D_1U_1 \) with all factors invertible, then \( L = L_1 \) and \( D = D_1 \) and \( U = U_1 \). "The three factors are unique."

Derive the equation \( L_1DU_1 = D_1U_1U_1^{-1} \). Are the two sides triangular or diagonal? Deduce \( L = L_1 \) and \( U = U_1 \) (they all have diagonal 's). Then \( D = D_1 \).

19. Triangular matrices have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into \( A = LU \) and \( A = LDU \).

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
a & a & 0 \\
a & a+b & b \\
0 & b & b+c
\end{bmatrix}
\]

20. When \( T \) is triangular, its \( L \) and \( U \) factors have only two nonzero diagonals. How would you take advantage of knowing the zeros in \( T \), in a code for Gaussian elimination? Find \( L \) and \( U \).

Tridiagonal

\[
T = \begin{bmatrix}
1 & 2 & 0 & 0 \\
2 & 3 & 1 & 0 \\
0 & 1 & 2 & 3 \\
0 & 0 & 3 & 4
\end{bmatrix}
\]

21. If \( A \) and \( B \) have nonzeros in the positions marked by \( x \), which zeros (marked by 0) stay zero in their factors \( L \) and \( U \)?

\[
A = \begin{bmatrix}
x & x & x \\
x & x & 0 \\
x & 0 & x \\
0 & 0 & x
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
x & x & 0 \\
x & 0 & x \\
0 & x & x \\
0 & x & x
\end{bmatrix}
\]

204

Chapter 2. Solving Linear Equations
2.7 Transposes and Permutations

We need one more matrix, and fortunately it is much simpler than the inverse. It is the "transpose" of A, which is denoted by $A^T$. The columns of $A^T$ are the rows of A.

When $A$ is an $m \times n$ matrix, the transpose is $n \times m$:

Transpose

$$ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}.$$

You can write the rows of $A$ into the columns of $A^T$. Or you can write the columns of $A$ into the rows of $A^T$. The matrix "flips over" its main diagonal. The entry in row $i$, column $j$ of $A^T$ comes from row $j$, column $i$ of the original $A$.

Exchange rows and columns

$$(A^T)_{ij} = A_{ji}.$$ 

The transpose of a lower triangular matrix is upper triangular. (But the inverse is still lower triangular.) The transpose of $A^T$ is $A$.

Note MATLAB's symbol for the transpose of $A$ is $A'$. Typing $[1 \ 2 \ 3]$ gives a row vector and the column vector is $v = [1 \ 2 \ 3]'$. To enter a matrix $M$ with second column $w = [4 \ 5 \ 6]'$ you could define $M = [v \ w]$. Quicker to enter by rows and then transpose the whole matrix: $M' = [1 \ 2 \ 3; 4 \ 5 \ 6]'$.

The rules for transposes are very direct. We can transpose $A + B$ to get $(A + B)^T$. Or we can transpose $A$ and $B$ separately, and then add $A^T + B^T$ with the same result.

The serious questions are about the transpose of a product $AB$ and an inverse $A^{-1}$:

$$ \text{Sum} \quad \text{The transpose of} \quad A + B \quad \text{is} \quad A^T + B^T. \quad (1) $$

$$ \text{Product} \quad \text{The transpose of} \quad AB \quad \text{is} \quad (AB)^T = B^T A^T. \quad (2) $$

$$ \text{Inverse} \quad \text{The transpose of} \quad A^{-1} \quad \text{is} \quad (A^{-1})^T = (A^T)^{-1}. \quad (3) $$

Notice especially how $B^T A^T$ comes in reverse order. For inverses, this reverse order was quick to check: $(B^{-1} A^{-1})^T = A^T B^T$, because $B^{-1} A^{-1}$ times $A B$ produces $I$. To understand $(AB)^T = B^T A^T$, start with $(Ax)^T = x^T A^T$.

$A^T$ combines the columns of $A$ while $x^T A^T$ combines the rows of $A^T$.

It is the same combination of the same vectors! In $A$ they are columns, in $A^T$ they are rows. So the transpose of the column $Ax$ is the row $x^T A^T$. That fits our formula $(Ax)^T = x^T A^T$.

Now we can prove the formula $(AB)^T = B^T A^T$, when $B$ has several columns.

If $B = [x_1 \ x_2]$ has two columns, apply the same idea to each column. The columns of $AB$ are $Ax_1$ and $Ax_2$. Their transposes are the rows of $B^T A^T$:

$$ \text{Transposing} \quad AB = \begin{bmatrix} Ax_1 & Ax_2 & \cdots \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1^T A^T \\ x_2^T A^T \\ \vdots \end{bmatrix} \quad \text{which is} \quad B^T A^T. \quad (4) $$
2.7. Transposes and Permutations

The right answer is \( B^T A^T \) comes out a row at a time. Here are numbers in \((AB)^T = B^T A^T:\)
\[
AB = \begin{bmatrix}
0 & 5 \\
1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
5 & 0 \\
9 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
5 & 4 \\
0 & 1 \\
\end{bmatrix}
\text{ and } \begin{bmatrix}
5 & 0 \\
9 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
5 & 0 \\
0 & 1 \\
\end{bmatrix}.
\]
The reverse order extends to three or more factors: \((ABC)^T = C^T B^T A^T\).

\[\text{If } A = LDU \text{ then } A^T = U^TD^TL^T. \quad \text{The pivot matrix has } D = D^T.\]

Now apply this product rule to both sides of \( A^{-1} A = I \). On one side, \( I^T \) is \( I \). We confirm the rule that \((A^{-1})^T \) is the inverse of \( A^T \), because their product is \( I \):
\[\text{Transpose of inverse } \quad A^{-1} A = I \quad \text{is transposed to } A^T(A^{-1})^T = I. \quad (5)\]

Similarly \( A^T = I \) leads to \((A^{-1})^T A^{-1} = I \). We can invert the transpose or we can transpose the inverse. Notice especially: \(A^T\) is invertible exactly when \(A\) is invertible.

Example 1 \quad The inverse of \( A = \begin{bmatrix}
3 & 4 \\
1 & 2 \\
\end{bmatrix}\) is \( A^{-1} = \begin{bmatrix}
-2 & 1 \\
1 & -1 \\
\end{bmatrix}\). The transpose is \(A^T = \begin{bmatrix}
3 & 1 \\
4 & 2 \\
\end{bmatrix}\).
\[
(A^T)^T \quad \text{and} \quad (A^{-1})^T \quad \text{are both equal to } \begin{bmatrix}
1 & -2 \\
-4 & 1 \\
\end{bmatrix}.
\]

The Meaning of Inner Products:

We know the dot product (inner product) of \(x\) and \(y\). It is the sum of numbers \(x_i y_j\).

Now we have a better way to write \(x \cdot y\), without using that unprofessional dot. Use matrix notation instead:

\[\text{If } \begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix} \text{ is inside } \begin{bmatrix} x^T \\ \end{bmatrix} \text{ the dot product or inner product is } x^T y = (1 \times n)(n \times 1) = n \times 1\]

\[\text{If } \begin{bmatrix} x^T \\ \end{bmatrix} \text{ is outside } \begin{bmatrix} y_1 \\ y_2 \\ \end{bmatrix} \text{ the rank one product or outer product is } x y^T = (n \times 1)(1 \times n)\]

\(x^T y\) is a number, \(x y^T\) is a matrix. Quantum mechanics would write these as \(<x|y>\) (inner) and \(|x><y|\) (outer). I think the world is governed by linear algebra, but physics disguises it well. Here are examples where the inner product has meaning:

From mechanics 
\[\text{Work (Motions) (Forces) } = x^T f\]

From circuits 
\[\text{Heat loss (Voltage drops) (Currents) } = e^T y\]

From economics 
\[\text{Income (Quantities) (Prices) } = q^T p\]

We are really close to the heart of applied mathematics, and there is one more point to explain. It is the deeper connection between inner products and the transpose of \(A\).

We defined \(A^T\) by flipping the matrix across its main diagonal. That's not mathematics. There is a better way to approach the transpose. \(A^T\) is the matrix that makes these two inner products equal for every \(x\) and \(y\):
\[(Ax)^T y = x^T (A^T y) \quad \text{Inner product of } Ax \text{ with } y = \text{inner product of } x \text{ with } A^T y\]

Example 2 \quad Start with \(A = \begin{bmatrix}
1 & 2 \\
2 & 1 \\
\end{bmatrix}\) \(x = \begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}\) \(y = \begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}\)

On one side we have \(Ax\) multiplying \(y^T = (x_2 - x_1)y_1 + (x_1 - x_2)y_2\)
That is the same as \(x_1(-y_1) + x_2(y_1 - y_2) + y_2(x_1 - x_2)\). Now \(x\) is multiplying \(A^T y\).
\[A^T y \text{ must be } \begin{bmatrix}
-y_1 \\
y_2 \\
\end{bmatrix}\]
which produces \(A^T = \begin{bmatrix}
-1 & 0 \\
1 & -1 \\
\end{bmatrix}\) as expected.

Example 3 \quad Will you allow me a little calculus? It is extremely important or I wouldn't leave linear algebra. (This is really linear algebra for functions \(x(t)\).) The [difference matrix changes to a derivative \(A = d/dt\). Its transpose will now come from \((dx/dt, y) = (x, -dy/dt)\).

The inner product changes from a finite sum of \(x_k y_k\) to an integral of \(x(t) y(t)\).

\[\text{Inner product of functions } \quad x^T y = \int_{\infty}^{-\infty} x(t) y(t) dt \quad \text{by definition}\]

\[\text{Transpose rule } \quad (Ax)^T y = x^T(A^T y) = \int_{\infty}^{-\infty} x(t) A^T y(t) dt = \int_{\infty}^{-\infty} x(t) -dy/dt dt \quad \text{shows } A^T = d/dt. \quad (6)\]

I hope you recognize "integration by parts." The derivative moves from the first function \(x(t)\) to the second function \(y(t)\). During that move, a minus sign appears. This tells us that the transpose of the derivative is minus the derivative.

The derivative is anti-symmetric: \(A = d/dt \quad A^T = -d/dt\). Symmetric matrices have \(A^T = A\), anti-symmetric matrices have \(A^T = -A\). In some way, the 2 by 3 difference matrix above followed this pattern. The 3 by 2 matrix \(A^T\) was minus a difference matrix. It produced \(y_1 - y_2\) in the middle component of \(A^T y\) instead of the difference \(y_2 - y_1\).

Symmetric Matrices

For a symmetric matrix, transposing \(A\) to \(A^T\) produces no change. Then \(A^T = A\). Its \((i, j)\) entry across the main diagonal equals its \((j, i)\) entry. In my opinion, these are the most important matrices of all.

**Definition** \quad A symmetric matrix has \(A^T = A\). This means that \(a_{ji} = a_{ij}\).

| Symmetric matrices \(A = \begin{bmatrix} 1 & 2 \\
2 & 1 \end{bmatrix}\) and \(D = \begin{bmatrix} 1 & 0 \\
0 & 10 \end{bmatrix}\) = \(D^T\). |

The inverse of a symmetric matrix is also symmetric. The transpose of \(A^{-1}\) is \((A^{-1})^T = A^{-1}\). That says \(A^{-1}\) is symmetric (when \(A\) is invertible):

**Symmetric inverses** \quad \(A^T = \begin{bmatrix} 1 & -2 \\
-2 & 2 \end{bmatrix}\) and \(D^{-1} = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix}\)\

Now we produce symmetric matrices by multiplying any matrix \(R \times R^T\).
Chapter 2. Solving Linear Equations

Symmetric Products $R^T R$ and $R R^T$ and $L D L^T$

Choose any matrix $R$, probably rectangular. Multiply $R^T$ times $R$. Then the product $R^T R$ is automatically a square symmetric matrix:

$$\text{The transpose of } R^T R \text{ is } R^T (R^T)^T \text{ which is } R^T R. \quad (7)$$

That is a quick proof of symmetry for $R^T R$. We could also look at the $(i, j)$ entry of $R^T R$. It is the dot product of row $i$ of $R^T$ (column $i$ of $R$) with column $j$ of $R$. The $(j, i)$ entry is the same dot product, column $j$ with column $i$. So $R^T R$ is symmetric.

The matrix $R R^T$ is also symmetric. (The shapes of $R$ and $R^T$ allow multiplication.) But $R R^T$ is a different matrix from $R^T R$. In our experience, most scientific problems that start with a rectangular matrix $R$ end up with $R^T R$ or $R R^T$ or both. As in least squares.

**Example 4** Multiply $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in both orders.

$$R R^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } R^T R = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

are both symmetric matrices.

The product $R^T R$ is $n$ by $n$. In the opposite order, $R R^T$ is $m$ by $m$. Both are symmetric, with positive diagonal (why?). But even if $m = n$, it is not very likely that $R^T R = R R^T$.

Equality can happen, but it is abnormal.

Symmetric matrices in elimination $A^T = A$ makes elimination faster, because we can work with half the matrix (plus the diagonal). It is true that the upper triangular $U$ is probably not symmetric. **The symmetry is in the triple product $A = L D U$.** Remember how the diagonal matrix $D$ of pivots can be divided out, to leave $1$’s on the diagonal of both $L$ and $U$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$LDU$ captures the symmetry

Now $U$ is the transpose of $L$.

When $A$ is symmetric, the usual form $A = L D U$ becomes $A = L D L^T$. The final $U$ (with $1$’s on the diagonal) is the transpose of $L$ (also with $1$’s on the diagonal). The diagonal matrix $D$ containing the pivots is symmetric by itself.

**If $A = A^T$ is factored into $L D U$ with no row exchanges, then $U$ is exactly $L^T$.**

**The symmetric factorization of a symmetric matrix is $A = L D L^T$.**

Notice that the transpose of $L D L^T$ is automatically $(L^T)^T D^T L^T$ which is $L D L^T$ again. The work of elimination is cut in half, from $n^3/3$ multiplications to $n^3/6$. The storage is also cut essentially in half. We only keep $L$ and $D$, not $U$ which is just $L^T$.

2.7. Transposes and Permutations

**Permutation Matrices**

The transpose plays a special role for a permutation matrix. This matrix $P$ has a single $1$ in every row and every column. Then $P^T$ is also a permutation matrix—maybe the same or maybe different. Any product $P_1 P_2$ is again a permutation matrix. We now create every $P$ from the identity matrix, by reordered the rows of $I$.

The simplest permutation matrix is $P = I$ (no exchanges). The next simplest are the row exchanges $P_{ij}$. These are constructed by exchanging two rows $i$ and $j$ of $I$. Other permutations reorder more rows. By doing all possible row exchanges to $I$, we get all possible permutation matrices:

**Definition** A permutation matrix $P$ has the rows of the identity $I$ in any order.

**Example 5** There are six $3$ by $3$ permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

There are $n!$ permutation matrices of order $n$. The symbol $n!$ means "in factorial," the product of the numbers $1(2) \cdots (n)$. Thus $3! = 1(2)(3)$ which is $6$. There will be $24$ permutation matrices of order $n = 4$. And $120$ permutations of order $5$.

There are only two permutation matrices of order $2$, namely $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**Important** $P^{-1}$ is also a permutation matrix. Among the six $3$ by $3$ matrices above, the four matrices on the left are their own inverses. The two matrices on the right are inverses of each other. In all cases, a single row exchange is its own inverse. If we repeat the exchange we are back to $I$. But for $P_3$, $P_{32}$, the inverses go in opposite order as always. The inverse is $P_3$, $P_{32}$.

More important: $P^{-1}$ is always the same as $P^T$. The two matrices on the right are transposes—and inverses—of each other. When we multiply $P P^T$, the "1" in the first row of $P$ hits the "1" in the first column of $P^T$ (since the first row of $P$ is the first column of $P^T$). It misses the ones in all the other columns. So $P P^T = I$.

Another proof of $P^T = P^{-1}$ looks at $P$ as a product of row exchanges. Every row exchange is its own transpose and its own inverse. $P^T$ and $P^{-1}$ both come from the product of row exchanges in reverse order. So $P^T$ and $P^{-1}$ are the same.

Symmetric matrices led to $A = L D L^T$. Now permutations lead to $PA = LU$. 


The PA = LU Factorization with Row Exchanges

We hope you remember A = LU. It started with A = (E_{11} \ldots E_{1n})U. Every elimination step was carried out by an E_{ij} and it was inverted by E_{ji}. Those inverses were compressed into one matrix L, bringing U back to A. The lower triangular L has 1's on the diagonal, and the result is A = LU.

This is a great factorization, but it doesn't always work. Sometimes row exchanges are needed to produce pivots. Then A = (E_1 \ldots E_{r-1} \ldots E_{r+1} \ldots E_n)U. Every row exchange is carried out by a P_{ij} and inverted by P_{ji}. We now comprress those row exchanges into a single permutation matrix P. This gives a factorization for every invertible matrix A—which we naturally want.

The main question is where to collect the P_{ij}'s. There are two good possibilities—

1. The row exchanges can be done in advance. Their product P puts the rows of A in the right order, so that no exchanges are needed for PA. Then PA = L U.

2. If we hold row exchanges until after elimination, the pivot rows are in a strange order. P_1 puts them in the correct triangular order in U_1. Then A = L_1 P_1 U_1.

PA = L U is constantly used in all computing (and in MATLAB). We will concentrate on this form. Most numerical analysts have never seen the other form.

The factorization A = L_1 P_1 U_1 might be more elegant. If we mention both, it is because the difference is not well known. Probably you will not spend a long time on either one. Please don't. The most important case has P = I, when A equals LU with no exchanges. For this matrix A, exchange rows 1 and 2 to put the first pivot in its usual place.

Then go through elimination on PA:

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 2 & 3 \\
2 & 7 & 9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
2 & 7 & 9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
0 & 3 & 7
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 & 4 \\
1 & 2 & 0 \\
0 & 1 & 3
\end{pmatrix}
\]

PA

The matrix PA has its rows in good order, and it factors as usual into L U:

\[
P = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix},
\]

PA = L U

where

\[
P A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix} = L U
\]

We started with A and ended with U. The only requirement is invertibility of A.

If A is invertible, a permutation P will put its rows in the right order to factor PA = LU

There must be a full set of pivots after row exchanges for A to be invertible.

2.7 Transposes and Permutations

In MATLAB, A([r k]) := A([lk r k]) exchanges row k with row r below it (where the kth pivot has been found). Then the lu code updates L and P and the sign of P:

A([r k]) := A([lk r k]);

This is part of

\[
\begin{pmatrix} I & U \end{pmatrix} = \text{lu}(A)
\]

\[
P([r k]) := P([lk r k]);
\]

sign := -sign

The “sign” of P tells whether the number of row exchanges is even (sign = +1). An odd number of row exchanges will produce sign = -1. At the start, P is I and sign = +1. When there is a row exchange, the sign is reversed. The final value of sign is the determinant of P and it does not depend on the order of the row exchanges.

For PA we get back to the familiar L U. This is the usual factorization. In reality, lu(A) often does not use the first available pivot. Mathematically we accept a small pivot, but not zero. It is better if the computer looks down the column for the largest pivot. (Section 9.1 explains why this “partial pivoting” reduces the roundoff error.) Then P may contain row exchanges that are not algebraically necessary. Still PA = L U.

Our advice is to understand permutations but let the computer do the work. Calculations of A = LU are enough to do by hand, without P. The Teaching Code splu(A) factors PA = L U and splu(A,b) solves Ax = b for any invertible A. The program splu stops if no pivot can be found in column k. Then A is not invertible.

- REVIEW OF THE KEY IDEAS -

1. The transpose puts the rows of A into the columns of A^T. Then (A^T)_{ij} = A_{ji}.

2. The transpose of AB is B^T A^T. The transpose of A^{-1} is the inverse of A^T.

3. The dot product is x \cdot y = x^T y. Then (A x) \cdot y equals the dot product x^T (A^T y).

4. When A is symmetric (A^T = A), its LDL factorization is symmetric: A = LDL^T.

5. A permutation matrix P has a 1 in each row and column, and P^T = P^{-1}.

6. There are n! permutation matrices of size n. Half even, half odd.

7. If A is invertible then a permutation P will reorder its rows for PA = LU.
### WORKED EXAMPLES

#### 2.7 A
Applying the permutation $P$ to the rows of $A$ destroys its symmetry:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \quad PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

What permutation $Q$ applied to the columns of $PA$ will recover symmetry in $PAQ$?

The numbers $1,2,3$ must come back to the main diagonal (not necessarily in order). Show that $Q$ is $P^T$, so that symmetry is saved by $PAQ = PAPT$.

**Solution**
To recover symmetry and put “2” back on the diagonal, column 2 of $PA$ must move to column 1. Column 3 of $PA$ (containing “3”) must move to column 2. Then the “1” moves to the 3,3 position. The matrix that permutes columns is $Q$:

$$PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad PAQ = \begin{bmatrix} 2 & 6 & 4 \\ 6 & 3 & 5 \\ 4 & 5 & 1 \end{bmatrix}$$

The matrix $Q$ is $P^T$. This choice always recovers symmetry, because $PAP^T$ is guaranteed to be symmetric. (Its transpose is again $PAP^T$.) The matrix $Q$ is also $P^{-1}$, because the inverse of every permutation matrix is its transpose.

If $D$ is a diagonal matrix, we are finding that $PDP^T$ is also diagonal. When $P$ moves row 1 down to row 3, $P^T$ on the right will move column 1 to column 3. The (1,1) entry moves down to (3,1) and over to (3,3).

#### 2.7 B
Find the symmetric factorization $A = LDL^T$ for the matrix $A$ above. Is this $A$ invertible? Find also the $PQ = LU$ factorization for $Q$, which needs row exchanges.

**Solution**
To factor $A$ into $LDL^T$ we eliminate below the pivots:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = U.$$  

The multipliers were $\ell_{31} = 4$ and $\ell_{32} = 5$ and $\ell_{33} = 1$. The pivots $1,-14,-8$ go into $D$.

When we divide the rows of $U$ by those pivots, $L^T$ should appear:

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ -14 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix $A$ is invertible because it has three pivots. Its inverse is $(L^T)^{-1}D^{-1}L^{-1}$ and $A^{-1}$ is also symmetric. The numbers 14 and 8 will turn up in the denominators of $A^{-1}$. The “determinant” of $A$ is the product of the pivots $1(-14)(-8) = 112$.

### Problem Set 2.7
Questions 1–7 are about the rules for transpose matrices.

1. Find $A^T$ and $A^{-1}$ and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}.$$  

2. Verify that $(AB)^T$ equals $B^TA^T$ but those are different from $A^TB^T$:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad AR = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$  

In case $AB = BA$ (not generally true!) how do you prove that $B^TA^T = A^TB^T$?
3 (a) The matrix \((AB)^{-1}\)T comes from \((A^{-1})^T\) and \((B^{-1})^T\). In what order?
(b) If \(U\) is upper triangular then \((U^{-1})^T\) is \(\boxed{\text{triangular}}\).

4 Show that \(A^2 = 0\) is possible but \(A^T A = 0\) is not possible (unless \(A = \text{zero matrix}\)).

5 (a) The row vector \(x^T\) times \(A\) times the column \(y\) produces what number?
\[
x^T Ay = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 10.
\]
(b) This is the row \(x^T A = \boxed{\text{times}}\) the column \(y = (0, 1, 0)\).
(c) This is the row \(x^T = \boxed{\text{times}}\) the column \(Ay = \boxed{\text{}}\).

6 The transpose of a block matrix \(M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is \(M^T = \boxed{\text{}}\). Test an example. Under what conditions on \(A, B, C, D\) is the block matrix symmetric?

7 True or false:
(a) The block matrix \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is automatically symmetric.
(b) If \(A\) and \(B\) are symmetric then their product \(A B\) is symmetric.
(c) If \(A\) is not symmetric then \(A^T\) is not symmetric.
(d) When \(A, B, C\) are symmetric, the transpose of \(ABC\) is \(CBA\).

Questions 8–15 are about permutation matrices.

8 Why are there \(n!\) permutation matrices of order \(n\)?

9 If \(P_1\) and \(P_2\) are permutation matrices, so is \(P_1 P_2\). This still has the rows of \(I\) in some order. Give examples with \(P_1 P_2 \neq P_2 P_1\) and \(P_3 P_4 = P_4 P_3\).

10 There are 12 "even" permutations of \((1, 2, 3, 4)\), with an even number of exchanges. Two of them are \((1, 2, 3, 4)\) with no exchanges and \((4, 3, 2, 1)\) with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, just order the numbers.

11 Which permutation makes \(PA\) upper triangular? Which permutations make \(P_1 A P_2\) lower triangular? Multiplying \(A\) on the right by \(P_2\) exchanges the \(\boxed{\text{}}\) of \(A\).

12 Explain why the dot product of \(x\) and \(y\) equals the dot product of \(Px\) and \(Py\). Then from \((Px)^T(Py) = x^T y\) deduce that \(P^T P = I\) for any permutation. With \(x = (1, 2, 3)\) and \(y = (1, 4, 2)\) choose \(P\) to show that \(P x \cdot y\) is not always \(x \cdot P y\).

13 (a) Find a 3 by 3 permutation matrix with \(P^3 = I\) (but not \(P = I\)).
(b) Find a 4 by 4 permutation \(P\) with \(P^4 \neq I\).

14 If \(P\) has 1's on the antidiagonal from \((1, n)\) to \((n, 1)\), describe \(PAP\). Note \(P = \boxed{P^T}\).

15 All row exchange matrices are symmetric: \(P^T = P\). Then \(P^T P = I\) becomes \(P^2 = I\). Other permutation matrices may or may not be symmetric.
(a) If \(P\) sends row 1 to row 4, then \(P^T\) sends row \(\boxed{\text{}}\) to row \(\boxed{\text{}}\).
When \(P^T = P\) the row exchanges come in pairs with no overlap.
(b) Find a 4 by 4 example with \(P^T = P\) that moves all four rows.

Questions 16–21 are about symmetric matrices and their factorizations.

16 If \(A = A^T\) and \(B = B^T\), which of these matrices are certainly symmetric?
(a) \(A^2 - B^2\) \(\quad\) (b) \((A + B)(A - B)\) \(\quad\) (c) \(ABA\) \(\quad\) (d) \(ABAB\).

17 Find 2 by 2 symmetric matrices \(A = A^T\) with these properties:
(a) \(A\) is not invertible.
(b) \(A\) is invertible but cannot be factored into \(LU\) (row exchanges needed).
(c) \(A\) can be factored into \(LDL^T\) but not into \(LL^T\) (because of negative \(D\)).

18 (a) How many entries of \(A\) can be chosen independently, if \(A = A^T\) is 5 by 5?
(b) How do \(L\) and \(D\) (still 5 by 5) give the same number of choices in \(LDL^T\)?
(c) How many entries can be chosen if \(A\) is skew-symmetric? (\(A^T = -A\)).

19 Suppose \(R\) is rectangular (\(m\) by \(n\)) and \(A\) is symmetric (\(m\) by \(m\)).
(a) Transpose \(R^T A R\) to show its symmetry. What shape is this matrix?
(b) Show why \(R^T R\) has no negative numbers on its diagonal.

20 Factor these symmetric matrices into \(A = LDL^T\). The pivot matrix \(D\) is diagonal:

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b & c \\ b & d \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.
\]

21 After elimination clears out column 1 below the first pivot, find the symmetric 2 by 2 matrix that appears in the bottom right corner:

\[
\text{Start from } A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & b & c \\ b & d \end{bmatrix}.
\]
23 Find a 4 by 4 permutation matrix (call it A) that needs 3 row exchanges to reach the end of elimination. For this matrix, what are its factors P, L, and U?

24 Factor the following matrix into \( PA = LU \). Factor it also into \( A = L_1 P_1 U_1 \) (hold the exchange of row 3 until 3 times row 1 is subtracted from row 2):

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11
\end{bmatrix}
\]

25 Extend the slu code in Section 2.6 to a code that factors \( PA = LU \).

26 Prove that the identity matrix cannot be the product of three row exchanges (or five). It can be the product of two exchanges (or four).

27 (a) Choose \( E_{21} \) to remove the 3 below the first pivot. Then multiply \( E_{21}AE_{12} \) to remove both 3's:

\[
A = \begin{bmatrix}
1 & 3 & 0 \\
3 & 1 & 4 \\
0 & 4 & 9
\end{bmatrix}
\]

(b) Choose \( E_{32} \) to remove the 4 below the second pivot. Then \( A \) is reduced to \( D \) by \( E_{32}E_{21}A \) is \( D \). Invert the \( E \)'s to find \( L \) in \( A = LDL^T \).

28 If every row of a 4 by 4 matrix contains the numbers 0, 1, 2, 3 in some order, can the matrix be symmetric?

29 Prove that no reordering of rows and reordering of columns can transpose a typical matrix. (Watch the diagonal entries.)

The next three questions are about applications of the identity \((Ax)^T y = x^T (A^T y)\).

30 Wires go between Boston, Chicago, and Seattle. Those cities are at voltages \( x_B, x_C, x_S \). With unit resistances between cities, the currents between cities are in \( y \):

\[
y = Ax \quad \text{is} \quad \begin{bmatrix}
x_B \\
x_C \\
x_S
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
x_B \\
x_C \\
x_S
\end{bmatrix}.
\]

(a) Find the total currents \( A^T y \) out of the three cities.

(b) Verify that \((Ax)^T y\) agrees with \(x^T (A^T y)\)—six terms in both.

2.7. Transposes and Permutations

31 Producing \( x_1 \) trucks and \( x_2 \) planes needs \( x_1 + 50x_2 \) tons of steel, 40\( x_1 + 1000x_2 \) pounds of rubber, and 2\( x_1 + 50x_2 \) months of labor. If the unit costs \( v_1, v_2, v_3 \) are $700 per ton, $3 per pound, and $3000 per month, what are the values of one truck and one plane? Those are the components of \( A^T y \).

32 \( Ax \) gives the amount of steel, rubber, and labor to produce \( x \) in Problem 31. Find \( A \). Then \( Ax + y \) is the number of inputs while \( x \cdot A^T y \) is the value of \( A^T y \).

33 The matrix \( P \) that multiplies \((x, y, z)\) to give \((c, x, y)\) is also a rotation matrix. Find \( P \) and \( P^T \). The rotation axis \( a = (1, 1, 1) \) doesn't move, it equals \( P \). What is the angle of rotation from \( v = (2, 3, -5) \) to \( Pv = (-5, 2, 3) \)?

34 Write \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) as the product \( EH \) of an elementary row operation matrix \( E \) and a symmetric matrix \( H \).

35 Here is a new factorization of \( A \) into \( PA\) times symmetric:

\[
\text{Start from } A = LDL^T. \quad \text{Then } A = L(U^T)^{-1} \text{ times } U^T \text{DU}.
\]

Why is \( (U^T)^{-1} \) triangular? Its diagonal is all 1's. Why is \( U^T \text{ symmetric} \)?

36 A group of matrices includes \( AB \) and \( A^{-1} \) if it includes \( A \) and \( B \). "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices \( L \) with 1's on the diagonal, symmetric matrices \( S \), positive matrices \( M \), diagonal invertible matrices \( D \), permutation matrices \( P \), matrices with \( QT = Q \).

\text{Invent two more matrix groups.}

\text{Challenge Problems}

37 A square northwest matrix \( B \) is zero in the southeast corner, below the antidiagonal that connects \((1, n)\) to \((n, 1)\). Will \( B^T \) and \( B^2 \) be northwest matrices? Will \( B^{-1} \) be northwest or southeast? What is the shape of \( BC = \text{northwest times southeast} \)?

38 If you take powers of a permutation matrix, why is some \( P^k \) eventually equal to \( I \)? Find a 5 by 5 permutation \( P \) so that the smallest power to equal \( I \) is \( P^6 \).

39 (a) Write down any 3 by 3 matrix \( A \). Split \( A \) into \( B + C \) where \( B = B^T \) is symmetric and \( C = -C^T \) is anti-symmetric.

(b) Find formulas for \( B \) and \( C \) involving \( A \) and \( A^T \). We want \( A = B + C \) and \( B = B^T \) and \( C = -C^T \).

40 Suppose \( Q^T \) equals \( Q^{-1} \) (transpose equals inverse, so \( Q^T Q = I \)).

(a) Show that the columns \( q_1, \ldots, q_n \) are unit vectors: \( \|q_i\|^2 = 1 \).

(b) Show that every two columns of \( Q \) are perpendicular: \( q_i^T q_j = 0 \).

(c) Find a 2 by 2 example with first entry \( q_1 = \cos \theta \).