For each problem, show your work. Your answers must be clear and follow a logical progression. Read the questions carefully. You do not need to compute decimal expansions of your answers. For example $\sin(8)$ and $\frac{23}{331}$ are very good answers.

Each problem is worth 15 points. **Answer 7 problems** for a total of 105 points. If you answer all 8, you can not score more than 105 points.

1. Find the root of $f(x) = e^x + x$ using one iteration of (A) Newton's Method and (B) fixed point iteration. (C) Justify your initial approximation.

(A) $f'(x) = e^x + 1$. Let $x_0 = -1$. Then using Newton's Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{e^{-1} - 1}{e^{-1} + 1} = \frac{-(e^{-1} + 1) - (e^{-1} - 1)}{e^{-1} + 1} = \frac{-2e^{-1}}{e^{-1} + 1}.$$

So $x_1 = -\frac{2}{1+e}$ is the approximation.

(B) Let $x_0 = -1$. We know that $f(x) = 0 \Leftrightarrow x = -e^x$ so we perform fixed point iteration with the function $g(x) = -e^x$:

$$x_1 = g(x_0) = -e^{-1} = -\frac{1}{e}.$$

(C) It is easy to check that $f(0) = 1 > 0$ and $f(-1) = \frac{1}{e} - 1 < 0$. Since f is continuous on $[-1, 0]$, the root must lie in $(-1, 0)$. Since $|f(-1) - 0| = \frac{1}{e} < 1 = |f(0) - 0|$, I chose the endpoint that is the better approximation to the root.

(If instead, you bisect this interval then $x_0 = -.5$ and (A) $x_1 = -\frac{3}{2+2\sqrt{e}}$ and (B) $x_1 = -\frac{1}{\sqrt{e}}$. Both are better respective approximations of the actual root.)

2. Let $g(x) = \frac{1}{x+2}$. Show that for any initial approximation, p_0, in the interval $[\frac{1}{4}, 1]$, the sequence generated by $p_{n+1} = g(p_n)$ will converge to a fixed point of f.

By the Fixed Point Theorem, I must show that for all $x \in [\frac{1}{4}, 1]$, $g(x) \in [\frac{1}{4}, 1]$ and that $g'(x)$ exists on $[\frac{1}{4}, 1]$ with $|g'(x)| \leq k < 1$ for all $x \in (\frac{1}{4}, 1)$.

(i) $g'(x) = \frac{-1}{(x+2)^2}$ exists on $(\frac{1}{4}, 1)$ and is negative on $[\frac{1}{4}, 1]$. Therefore, g is monotonically decreasing on $[\frac{1}{4}, 1]$. Since $g(1) = \frac{1}{3} > \frac{1}{4}$ and $g(\frac{1}{4}) = \frac{4}{9} < 1$, then $g(x) \in [\frac{1}{4}, 1]$ for all $x \in [\frac{1}{4}, 1]$.

(ii) Let $h(x) = |g'(x)| = \frac{1}{(x+2)^2}$. Then $h'(x) = \frac{-2}{(x+2)^3} < 0$ on $[\frac{1}{4}, 1]$. Since h is monotonically decreasing on $[\frac{1}{4}, 1]$, we have

$$\left|g'(x)\right| = h(x) \leq h\left(\frac{1}{4}\right) = \frac{1}{\left(\frac{9}{4}\right)^2} = \frac{16}{81} < 1.$$

So let $k = \frac{16}{81}$.

Thus, by the Fixed Point Theorem, for any $p_0 \in [\frac{1}{4}, 1]$, the sequence $p_{n+1} = g(p_n)$ converges to a unique fixed point $p = g(p) \in [\frac{1}{4}, 1]$.

3. Suppose the sequence p_n converges to p of order α with asymptotic error constant $\lambda > 0$. Prove that for all $\beta < \alpha$, p_n converges to p of order β and determine the appropriate asymptotic error constant.

Since $p_n \to p$ of order α with asymptotic error constant λ, then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$

Let $\beta < \alpha$. Then $\alpha - \beta > 0$ and since $p_n \to p$, we have that $\lim_{n\to\infty} |p_n - p|^{\alpha-\beta} = 0$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\beta} &= \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha \, |p_n - p|^{\beta-\alpha}} \\
&= \left(\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}\right)\left(\lim_{n\to\infty} \frac{1}{|p_n - p|^{\beta-\alpha}}\right) \\
&= \lambda \left(\lim_{n\to\infty} |p_n - p|^{\alpha-\beta}\right) \\
&= \lambda\,(0) \\
&= 0.
\end{aligned}$$

Hence, $p_n \to p$ of order β with asymptotic error constant 0.

4. Suppose $f(x) \in C^3[0,2]$ and that $\left|f^{(3)}(x)\right| \leq 3$ for all $x \in [0,2]$. Also suppose $f(.5) = 1$, $f(1) = 2$, and $f(1.25) = 4$.
(A) Determine the Langrange polynomial that interpolates f at the points $x_0 = .5$, $x_1 = 1$, and $x_2 = 1.25$.

(B) Determine the best upper bound on the error when you approximate $f(.75)$ with your Lagrange polynomial.

(A) $$\begin{aligned}
P_2(x) &= 1\frac{(x-1)(x-1.25)}{(-.5)(-.75)} + 2\frac{(x-.5)(x-1.25)}{(.5)(-.25)} + 4\frac{(x-.5)(x-1)}{(.75)(.25)} \\
&= 8x^2 - 10x + 4.
\end{aligned}$$

(B) $$|f(x) - P_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(x-.5)(x-1)(x-1.25)\right| \leq \frac{1}{2}\,|(x-.5)(x-1)(x-1.25)|$$

with the last inequality coming from the bound given in the problem. Therefore, we have

$$|f(.75) - P_2(.75)| \leq \frac{1}{2}(.25)(.25)(.5) = \frac{1}{2\cdot 4\cdot 4\cdot 2} = \frac{1}{64}.$$

5. Suppose $H(x)$ is the Hermite polynomial determined from the following data:

x	$f(x)$	$f'(x)$
1	1	.5
1.3	2.3	1.5
1.5	3.8	-.5
1.8	2.4	.75
2	3	2

(A) What is the maximum degree of $H(x)$?
(B) What are the values $H(1.8)$ and $H'(1.3)$?
(C) If $f \in C^{\infty}[1,2]$, what is $|f(x) - H(x)|$ for any $x \in (1,2)$?

(A) A Hermite polynomial interpolating $n+1$ nodes has degree at most $2n+1$ so the degree of H is at most $2(4)+1=9$.

(B) Hermite polynomials agree with f and f' at each of the nodes. Thus, $H(1.8) = f(1.8) = 2.4$ and $H'(1.3) = f'(1.3) = 1.5$.

(C) For all $x \in (1,2)$, the error bound is

$$|f(x) - H(x)| = \frac{\left|f^{(10)}(\xi)\right|}{10!}(x-1)^2(x-1.3)^2(x-1.5)^2(x-1.8)^2(x-2)^2$$

where $\xi = \xi(x) \in (1,2)$.

6. Suppose g is a function defined on $[a,b]$ and $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a set of nodes in $[a,b]$. Describe the properties of the clamped cubic spline interpolant of g. What is the main advantage of cubic spline interpolation?

The clamped cubic spline, $S(x)$, is a piecewise cubic polynomial interpolating g at the nodes and satisfying the following conditions:
(a) $S(x) = S_i(x)$ on $[x_i, x_{i+1}]$ for $i = 0, \ldots, n-1$ where $S_i(x)$ is a cubic polynomial;
(b) $S_i(x_i) = f(x_i)$ and $S_i(x_{i+1}) = f(x_{i+1})$ for $i = 0, \ldots, n-1$;
(c) $S_i(x_{i+1}) = S_{i+1}(x_{i+1})$ for $i = 0, \ldots, n-2$;
(d) $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ for $i = 0, \ldots, n-2$;
(e) $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$ for $i = 0, \ldots, n-2$;
(f) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$.

The main advantage of a cubic spline interpolating polynomial is that $S \in C^2[a,b]$ so it is a smooth function. In fact, we saw in class that S satisfies a minimal curvature condition with respect to the curvature of f.

7. (A) Suppose M is an upper bound on the second derivative of the function f on the interval $[a, b]$. Let $\tilde{f}(x)$ be your computer approximation to the function f where roundoff error only guarantees that $\left|f(x) - \tilde{f}(x)\right| < \epsilon$. Then the error term for the forward difference formula for approximating $f'(x)$, $x \in [a, b]$, is bounded above by $\frac{2\epsilon}{h} + \frac{h}{2}M$. Using this bound, find the optimal step size, h, for the forward difference formula to approximate $f'(x)$.

(B) If $f(x) = x^3 + x$, use a 3-point formula with $h = .5$ to approximate $f'(1.5)$.

(A) This error bound depends on the step size h: $e(h) = \frac{2\epsilon}{h} + \frac{h}{2}M$. So the optimal step size will minimize this error. We use basic calculus to find the minimizer of e:

$$e'(h) = \frac{-2\epsilon}{h^2} + \frac{M}{2} = 0 \qquad \Leftrightarrow \qquad \frac{2\epsilon}{h^2} = \frac{M}{2} \qquad \Leftrightarrow \qquad h^2 = \frac{4\epsilon}{M}.$$

So $h = \pm 2\sqrt{\epsilon/M}$ are the critical points. Since $e''(h) = \frac{4\epsilon}{h^3} > 0$ when $h > 0$, then $h = 2\sqrt{\epsilon/M}$ is the minimizer of $e(h)$. Thus the optimal step size is $h = 2\sqrt{\epsilon/M}$.

(B) The centered 3-point formula for approximating a derivative is

$$f'(x) \approx \frac{1}{2h}\left[f(x+h) - f(x-h)\right].$$

So with $f(x) = x^3 + x$ and $h = .5$, we have

$$f'(1.5) \approx \frac{1}{2(.5)}\left[(8+2) - (1+1)\right] = 8.$$

8. (A) Use Simpson's method (not Composite Simpson's, just Simpson's) to approximate

$$\int_0^2 \frac{x-1}{x^3+1}dx.$$

(B) Find the minimum n required for Composite Simpson's method to approximate the following integral with error less than .01:

$$\int_1^2 \ln(x)dx.$$

You may use that $f(x) = \ln(x)$ then $\left|f^{(4)}(x)\right| \leq 6$ on $[1,2]$.

(A) Simpson's Method is $\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$ where $h = \frac{b-a}{2}$. Thus,

$$\int_0^2 \frac{x-1}{x^3+1}dx \approx \frac{1}{3}\left[\frac{-1}{1} + 4\left(\frac{0}{2}\right) + \frac{1}{9}\right] = \frac{1}{3}\left[-\frac{9}{9} + \frac{1}{9}\right] = -\frac{8}{27}.$$

(B) In Composite Simpson's method, $h = \frac{b-a}{n}$ and the absolute error term is

$$\frac{b-a}{180}h^4\left|f^{(4)}(\xi)\right| = \frac{(b-a)^5}{180}\frac{1}{n^4}\left|f^{(4)}(\xi)\right|.$$

With $f(x) = \ln(x)$, then $\left|f^{(4)}(x)\right| \leq 6$ on $[1,2]$. For this integral, $b-a = 1$. Therefore, we must find n so that

$$\frac{1}{180}\frac{1}{n^4}6 < .01 = \frac{1}{100}.$$

Solving for n we have

$$\frac{1}{n^4} < \frac{3}{10} \qquad \Leftrightarrow \qquad n^4 > \frac{10}{3} \qquad \Leftrightarrow \qquad n > \sqrt[4]{\frac{10}{3}}.$$

We know $2 = \sqrt{4} > \sqrt{\frac{10}{3}} > \sqrt[4]{\frac{10}{3}}$. Then $n = 2$ is sufficient since we need the smallest even integer n satisfying $n > \sqrt[4]{\frac{10}{3}}$.