Solutions to Homework 6

View this online to see color in the figures.

Section 8.1

8. I input the data to Matlab, created the sums from matlab, and determined the appropriate coefficients:

$$a_0 = -.8028 \qquad \text{and} \qquad a_1 = .2233.$$

So the linear least squares approximation for this data is

$$\text{TestScore} = .2233(\text{HomeworkScore}) - .8028.$$

Using this equation, a homework score of 406.6404 predicts a test score of 90 while a homework score of 272.2920 predicts a test score of 60. Figure 1 depicts the linear least squares approximation and the data points. (Remeber what I presented after the midterm? We had a very strong linear fit.)

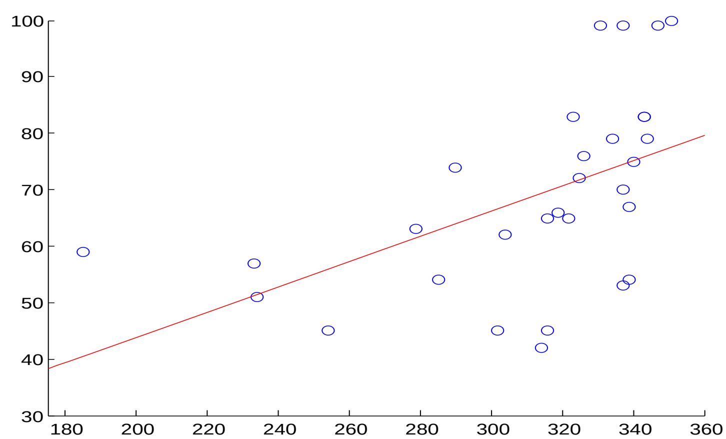

Figure 1: The linear least squares approximation $LSA(x) = .2233x - .8028$ of the plotted data.

G14) In class I formulated this matrix equation as given in the hint: $A = (a_{ij})_{1 \le i,j \le n+1}$ with $a_{ij} = \sum_{k=0}^{m} x_k^{i+j-2}$ with x_k distinct for $k = 1, \ldots, m$ where $n < m - 1$. Clearly, $a_{ij} = a_{ji}$ so A is symmetric. The hint tells us exactly how to prove that A is nonsingular. Seeking a contradiction, suppose A is singular and let c be a vector of size $(n+1) \times 1$ such that $c \neq 0$ and $c^t A c = 0$. Let $\vec{x}$ be the $(n+1) \times 1$ vector with $\vec{x}_i = x^{i-1}$ and define the polynomial $P(x) = c^t \vec{x}$. Then $\deg(P) \le n$ and therefore P has at most n roots. Since $c \neq 0$, it must be that $c^t A = 0$. However, the jth column of A is $A_j = (\sum_{k=1}^{m} x_k^{j-1}, \sum_{k=1}^{m} x_k^{j}, \ldots, \sum_{k=1}^{m} x_k^{j+n-1})^t$. Therefore we have $c^t A_j = P\left(\sum_{k=1}^{m} x_k^{j-1}\right) = 0$. Therefore, P has at least $n+1$ roots, namely $m, \sum_{k=1}^{m} x_k, \sum_{k=1}^{m} x_k^2, \ldots, \sum_{k=1}^{m} x_k^n$ which is a contradiction. Thus, A is nonsingular.

Section 8.2

8d) We first complete exercise 7b to get the orthogonal polynomials via the Gram-Schmidt process. They are:

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x - 1$$
$$\varphi_2(x) = x^2 - 2x + \frac{2}{3}$$
$$\varphi_3(x) = x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}.$$

(To obtain these polynomials, we must compute $B_1 = B_2 = B_3 = 1, C_2 = \frac{1}{3}, C_3 = \frac{4}{15}$ which I leave to you.)

Now we use these orthogonal polynomials to complete exercise 2d for $f(x) = e^x$ on $[0, 2]$. We compute $\alpha_k = \int_0^2 [\varphi_k(x)]^2 dx$ to obtain $\alpha_0 = 2$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{8}{45}$, and $\alpha_3 = \int_0^2 (x^3 - 3x^2 + (12/5)x - 2/5)^2 dx = \frac{8}{175}$. ($\alpha_i$ for $i = 0, 1, 2$ we already computed as denominators in the Gram-Schmidt process.) Finally, we must compute the coefficients a_k in the polynomial $P(x) = \sum_0^3 a_k\varphi_k(x)$ which, according to Theorem 8.6, is the least squares approximation to $f(x) = e^x$ on $[0, 2]$. For this problem, $a_k = \frac{1}{\alpha_k}\int_0^2 \varphi_k(x)e^x dx$.

$$a_0 = \frac{1}{2}\int_0^2 e^x dx = \frac{1}{2}\left(e^2 - 1\right) \approx 3.1945$$
$$a_1 = \frac{3}{2}\int_0^2 (x-1)e^x dx = \frac{3}{2}(2) = 3$$
$$a_2 = \frac{45}{8}\int_0^2 \left(x^2 - 2x + \frac{2}{3}\right)e^x dx = \frac{45}{8}\left(-\frac{14}{3} + \frac{2}{3}e^2\right) \approx 1.4590$$
$$a_3 = \frac{175}{8}\int_0^2 \left(x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}\right)e^x dx = \frac{175}{8}\left(\frac{74}{5} - 2e^2\right) \approx .4788.$$

Bonus 14) Suppose $\{\varphi_k\}_{k=0}^n$ is w-orthogonal on $[a, b]$. Suppose that $\sum_{k=0}^n c_k\varphi_k(x) = 0$ for all $x \in [a, b]$. Then for any $j \in \{0, 1, \ldots, n\}$, $\left(\sum_{k=0}^n c_k\varphi_k(x)\right)\varphi_j(x) = 0$ on $[a, b]$. Therefore, we have

$$0 = \int_a^b w(x)\left(\sum_{k=0}^n c_k\varphi_k(x)\right)\varphi_j(x)dx = \sum_{k=0}^n c_k\int_a^b w(x)\varphi_k(x)\varphi_j(x)dx$$
$$= c_j\int_a^b w(x)\left[\varphi_j(x)\right]^2 dx = c_j\alpha_j.$$

Since $\alpha_j > 0$, it must be that $c_j = 0$. Therefore, $c_j = 0$ for every $j = 0, \ldots, n$ which implies that $\{\varphi_k\}_{k=0}^n$ is a linearly independent set of functions on $[a, b]$.

Section 8.3

6) The degree five Maclaurin polynomial for e^x is $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$. So the degree six Maclaurin polynomial for xe^x is

$$P_6(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$$

with maximum error

$$\max_{x\in[-1,1]} |R_6(x)| \le \frac{\max_{x\in[-1,1]} |(7+x)e^x|}{7!} \le \frac{8e}{5040} \approx .0043.$$

To reduce the degree of this polynomial using Chebyshev polynomials, we need $\tilde{T}_6(x)$. The text lists the first four Chebyshev polynomials. Using the functional relationship of the Chebyshev polynomials, we have:

$$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 2xT_5(x) - T_4(x) = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1.$$

Therefore we have $\tilde{T}_6(x) = x^6 - \frac{3}{2}x^4 + \frac{9}{16}x^2 - \frac{1}{32}$. Following the method for economization of power series outlined in the text and in class we construct the polynomial

$$P_5(x) = P_6(x) - \frac{1}{120}\tilde{T}_6(x) = \frac{1}{24}x^5 + \frac{43}{240}x^4 + \frac{1}{2}x^3 + \frac{1911}{1920}x^2 + x - \frac{1}{3840}.$$

We know that the error bound is now $.0043 + \max|P_6(x) - P_5(x)| = .0043 + \frac{1}{120}\cdot\frac{1}{2^5} < .0043 + .0003 = .0046$. Still within our tolerance, we use the economization technique again and obtain

$$P_4(x) = P_5(x) - \frac{1}{24}\tilde{T}_5(x) = \frac{43}{240}x^4 + \frac{53}{96}x^3 + \frac{637}{640}x^2 + \frac{379}{384}x - \frac{1}{3840}.$$

Since $|P_5(x) - P_4(x)| = \frac{1}{24}\cdot\frac{1}{16} < .0027$, we know that $|e^x - P_4(x)| < .0073$ is still within our error tolerance of .01. If we reduce the polynomial one more time, the error term $|P_4(x) - P_3(x)| = \frac{43}{240}\left|\tilde{T}_4(x)\right| = \frac{43}{240}\cdot\frac{1}{8} \approx .0224$ and this will not keep us within our tolerance of .01. Thus, $P_4(x)$ is the least degree polynomial approximation of xe^x using the economization of power series with monic Chebyshev polynomials. See Figure 2 below.

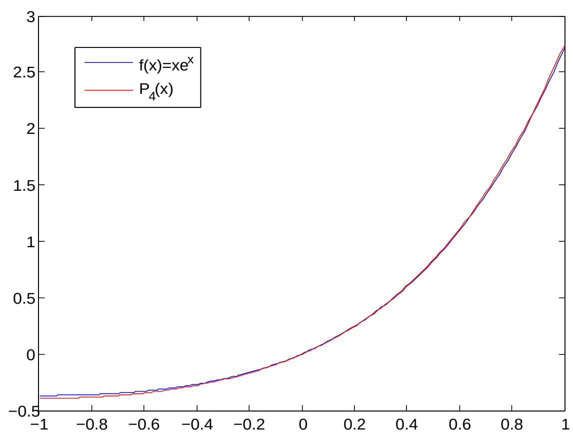

Figure 2: The plots of $f(x) = xe^x$ and $P_4(x)$. (The scaling of the axes is off to save space.)

Section 8.5

4) We seek the continuous least squares trigonometric polynomial for $f(x) = e^x$. From Theorem 8.6 and page 524, we have

$$S_n(x) = \frac{a_0}{2} + a_n\cos(nx) + \sum_{k=1}^{n-1}\left[a_k\cos(kx) + b_k\sin(kx)\right]$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(kx)dx$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(kx)dx.$$

To evaluate these integrals, we could use an integral table. But it is certainly cooler to derive these integrals with the handy integration by parts technique from calculus. Here I show the derivation for the integral in the definition of a_k. We use integration by parts on the integral involving $\cos(kx)$ which results in the integral involving $\sin(kx)$. Using integration by parts on this integral, we then solve for our original integral:

$$\begin{aligned}\int e^x\cos(kx)dx &= e^x\cos(kx) + k\int e^x\sin(kx)\\ &= e^x\cos(kx) + ke^x\sin(kx) - k^2\int e^x\cos(kx).\end{aligned}$$

Therefore, solving for $\int e^x\cos(kx)dx$ we have

$$\int e^x\cos(kx)dx = \frac{e^x}{1+k^2}\left[\cos(kx) + k\sin(kx)\right].$$

Using this identity, we evaluate the definite integral to obtain

$$a_k = \frac{1}{\pi(1+k^2)}(e^\pi - e^{-\pi})\cos(k\pi) = \frac{(-1)^k(e^\pi - e^{-\pi})}{\pi(1+k^2)}.$$

Performing a similar set of calculations, we obtain

$$b_k = \frac{k(-1)^{k+1}(e^\pi - e^{-\pi})}{\pi(1+k^2)} = \frac{(-1)^k(e^\pi - e^{-\pi})}{\pi(1+k^2)}(-k).$$

Therefore, we have

$$\begin{aligned}S_n(x) &= \frac{e^\pi - e^{-\pi}}{2\pi} + \frac{(-1)^n(e^\pi - e^{-\pi})}{\pi(1+n^2)}\cos(nx) + \sum_{k=1}^{n-1}\frac{(-1)^k(e^\pi - e^{-\pi})}{\pi(1+k^2)}\left[\cos(kx) - k\sin(kx)\right]\\ &= \frac{\sinh\pi}{\pi}\left[1 + 2\frac{(-1)^n}{1+n^2}\cos(nx) + 2\sum_{k=1}^{n-1}\frac{(-1)^k}{1+k^2}\left[\cos(kx) - k\sin(kx)\right]\right]. \qquad (1)\end{aligned}$$

This problem is a quasi-proof that calculus and Fourier series are totally awesome. If you don't think that was cool, keep doing the problem over and over until you do! Seriously, the limit of (1) as $n \to \infty$ is the exponential function!

16) The Fourier series for $f(x) = |x|$ is

$$\begin{aligned} S(x) &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2} \cos(kx) \\ &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2} \cos[(2k+1)x] \end{aligned}$$

since $(-1)^{2k+1} - 1 = -2$ and $(-1)^{2k} - 1 = 0$. With the assumption that $S(0) = f(0) = 0$, we observe that

$$0 = S(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

So, we obtain the value of the convergent sum (a classical result):

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$