Solutions to Homework 1

Section 1.1

18) The nth Taylor polynomial for $f(x) = \frac{1}{1-x}$ centered at 0:
$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$ so $f^{(k)}(0) = k!$. Hence

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} x^k.$$

To find a value of n necessary to approximate $f(x)$ with tolerance 10^{-6} on $[0, .5]$, we can not use the residual formula from Taylor's theorem as this produces an upper bound of 1. Instead, we recognize the Taylor series for $f(x)$ as the sum of a geometric series, with the nth partial sum being P_n. Therefore, we need to bound the residual formulated as the tail of the power series for $f(x) = \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k$:

$$R_n(x) = \sum_{k=n+1}^{\infty} x^k$$

which is a geometric series with constant x^{n+1} and ratio x. Therefore

$$R_n(x) = \sum_{k=n+1}^{\infty} x^k = x^{n+1}\sum_{k=0}^{\infty} x^k = \frac{x^{n+1}}{1-x}.$$

Therefore, on $[0, .5]$ we find the residual bounded as follows:

$$\frac{x^{n+1}}{1-x} \leq \frac{\left(\frac{1}{2}\right)^{n+1}}{1-x} \leq \frac{\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 2^{-n}.$$

Thus we find n such that $R_n(x) \leq 2^{-n} < 10^{-6}$ or $n > 6\log_2(10) \approx 19.9$. So $n = 20$ is sufficient.

22)(a) $P_n(x)$ "best" approximates $f(x)$ near x_0 because the first n derivatives of both f and P_n are identical at x_0. Therefore, in some interval around x_0, $P_n(x)$ will be a very good approximation of f. In fact, the higher derivatives will cause the graphs of the two functions to be nearly identical in an interval, I_n, about x_0 where the length of I_n grows with n (or at least is non-decreasing with n).

(b) If the tangent line to $f(x)$ at $x_0 = 1$ is $y = 4x - 1$, then $f'(x_0) = f'(1) = 4$. We are given $f''(x) = 6$. The quadratic polynomial that best approximates $f(x)$ near 1 is the Taylor polynomial of order 2 centered at 1. Thus $f(x) \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$. The only missing term is $f(1)$ but we know the tangent line at $x = 1$ which touches the graph at $(1, f(1))$. Therefore $f(1) = 3$ and we have

$$f(x) \approx 3 + 4(x-1) + 3(x-1)^2 = 3x^2 - 2x + 2.$$

G26) Let $k = \frac{c_1}{c_1+c_2}f(x_1) + \frac{c_2}{c_1+c_2}f(x_2)$ and define $\alpha = \frac{c_1}{c_1+c_2}$ so that $(1-\alpha) = \frac{c_2}{c_1+c_2}$. Since $c_1, c_2 > 0$, then $\alpha \in (0,1)$. Now $k = \alpha f(x_1) + (1-\alpha)f(x_2) = f(x_2) - \alpha\left[f(x_2) - f(x_1)\right]$. Since $\alpha \in (0,1)$, we have that k is between $f(x_1)$ and $f(x_2)$. f is continuous on $[a, b]$ and therefore is also continous on the interval defined by x_1 and x_2 which is contained in $[a, b]$. Invoking the intermediate value theorem, there exists ξ between x_1 and x_2 such that $f(\xi) = k$ which completes the proof.

Section 1.2

12) (a) With direct substitution we obtain the indeterminant form $\frac{0}{0}$ so we employ L'hopital's rule:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{1}=2.$$

(b) $f(.1)=\frac{1.11-.905}{.100}=2.05$

(c) e^x has Maclaurin series $\sum_{n=1}^{\infty}\frac{1}{n!}x^n$ so $P_3(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$. With three digit rounding we obtain

$$P_3(.1)=1.00+.100+.00500+.000167=1.11$$
$$P_3(-.1)=1.00-.100+.00500-.000167=.905$$

so $f(.1)=\frac{1.11-.905}{.100}=2.05$.

(d) The relative error is $\frac{2.05-2.003335}{2.003335}\approx .0233$.

22) The nearest centimeter means the measurements fall into intervals:

$$3\in[2.5,3.5)\qquad 4\in[3.5,4.5)\qquad 5\in[4.5,5.5).$$

Hence the upper bound on the volume is the product of the largest possible values, $3.5*4.5*5.5=86.625$ cm^3. Likewise the lower bound on volume is $2.5*3.5*4.5=39.375$ cm^3. The upper and lower bounds for the surface area are

$$2\,(3.5*4.5+3.5*5.5+4.5*5.5)=119.5\text{ cm}^2\quad\text{and}$$
$$2\,(3.5*4.5+3.5*2.5+4.5*2.5)=71.5\text{ cm}^2.$$

Section 1.3

6) The Maclaurin series for $\sin x$ is $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$. We use this in the following parts.

(a) $\sin(\frac{1}{n})=\frac{1}{n}-\frac{1}{3!n^3}+\frac{1}{5!n^5}-\cdots$ so $\sin\frac{1}{n}=0+O(\frac{1}{n})$.

(b) $\sin(\frac{1}{n^2})=\frac{1}{n^2}-\frac{1}{3!n^6}+\frac{1}{5!n^{10}}-\cdots$ so $\sin\frac{1}{n^2}=0+O(\frac{1}{n^2})$.

(c) $\left[\sin(\frac{1}{n})\right]^2=0+O(\frac{1}{n^2})$ because

$$\begin{aligned}\left[\sin\left(\frac{1}{n}\right)\right]^2&=\left[\frac{1}{n}-\frac{1}{3!n^3}+\frac{1}{5!n^5}-\cdots\right]*\left[\frac{1}{n}-\frac{1}{3!n^3}+\frac{1}{5!n^5}-\cdots\right]\\&=\left[\frac{1}{n^2}-\frac{2}{3!n^4}+\cdots\right].\end{aligned}$$

(d) The Taylor series for $\ln x$ centered at 1 is

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}.$$

Also $\ln(n+1)-\ln(n)=\ln(\frac{n+1}{n})=\ln(1+\frac{1}{n})$. So for large n we have

$$\ln(1+\frac{1}{n})\approx\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{1}{n}\right)^3-\frac{1}{4}\left(\frac{1}{n}\right)^4+\cdots$$

which clearly confirms $\lim_{n\to\infty}[\ln(n+1)-\ln(n)]=0$ and we have $\ln(n+1)-\ln(n)=0+O(\frac{1}{n})$.

8) (a) For each i there are i multiplications in $\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j$ since j ranges from 1 to i. There are exactly n iterations so there are a total of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ multiplications. There are $i-1$ additions for each i so there are a total of $\sum_{i=1}^{n}(i-1)=\frac{n(n-1)}{2}$ additions at each i. Then we must add these sums together for $n-1$ more additions. Therefore, there are n^2+n-1 total computations.

(b) We can reduce the number of computations by writing this double sum in the form

$$\sum_{i=1}^{n}\left(a_i \sum_{j=1}^{i} b_j\right).$$

Now there are still $i-1$ additions for each i but only one multiplication. Then we must sum these terms resulting in $n-1$ more additions. For this formulation there are $\frac{n(n-1)}{2}+n+n-1=\frac{1}{2}(n^2+3n-1)$ which is a reduction for $n>1$.

10) INPUT a, b, c (coefficients of a quadratic ax^2+bx+c)
OUTPUT x_1, x_2 (roots of the quadratic)
If $b=0$
then $x_1=\sqrt{\frac{-c}{a}}, x_2=-\sqrt{\frac{-c}{a}}$.
else if $b>0$
then $x_1=-\frac{b+\sqrt{b^2-4ac}}{2a}, x_2=-\frac{2c}{b+\sqrt{b^2-4ac}}$
else $x_1=-\frac{-2c}{b-\sqrt{b^2-4ac}}, x_2=\frac{b-\sqrt{b^2-4ac}}{2a}$
end
end
OUTPUT(x_1, x_2). STOP.

NOTE: There was some confusion on this problem. The text was asking about choosing the best way to compute the output. If this was implemented in Matlab, for instance, you wouldn't need to tell it to compute an imaginary number as this would happen automatically. If you did this in your psuedocode, that was fine. The key idea was when we should use the formulations based on sgn(b).

Section 2.1

20) We know $1.7=x(1)=\frac{-g}{2w^2}\left(\frac{e^w-e^{-w}}{2}-\sin w\right)$. Letting $f(w)=\frac{-32.17}{2w^2}\left(\frac{e^w-e^{-w}}{2}-\sin w\right)-1.7$ we know that $f(-1)=3.67>0$ and $f(-.1)=-1.6<0$. Therefore we utilize the bisection method with intial interval $[-1,-.1]$. My algorithm provides the estimate $p=-.317056$ with error less than or equal to .000003.

Section 2.2

8) $g(x)=2^{-x}$ is continuous on $[\frac{1}{3},1]$. $g'(x)=-(\ln 2)2^{-x}$ and g is therefore decreasing on $[\frac{1}{3},1]$. Since $g(1)=.5$ and $g(\frac{1}{3})\approx .7937$, we have that for all $x\in[\frac{1}{3},1]$, $.5\le g(x)<.8$ so that $g:[\frac{1}{3},1]\to[\frac{1}{3},1]$. Also, $|g'(x)|=(\ln 2)\,2^{-x}$ is also decreasing on $[\frac{1}{3},1]$. Thus for all $x\in[\frac{1}{3},1]$, $|g'(x)|\le(\ln 2)\,2^{-\frac{1}{3}}$. Therefore, by Theorem 2.2, there is a unique fixed point in $[\frac{1}{3},1]$. My algorithm with tolerance 10^{-4} and initial approximation $p_0=.5$ returns $p=.6412$.

With my choice of $p_0 = .5$ and $k = \frac{\ln 2}{2^{\frac{1}{3}}}$, Cor. 2.4 provides

$$|p_n - p| \leq \left(\frac{\ln 2}{2^{\frac{1}{3}}}\right)^n \left(\frac{1}{2}\right).$$

So we want n such that $\left(\frac{\ln 2}{2^{\frac{1}{3}}}\right)^n \left(\frac{1}{2}\right) < .0001$. Using the properties of logarithms, we solve to see that $n > 14.253$ is necessary. Therefore, we can expect an upper bound of 15 iterations.

Since the theoretical upperbound on the error is a worst case scenario, we anticipate that in practice we will have smaller error. This should require us to need fewer iterations of the method. In fact, my algorithm achieved the accuracy .0001 in 11 iterations.

20) Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1$$

converges to $\sqrt{A}$ whenver $x_0 > 0$.

Let $g(x) = \frac{1}{2}\left(x + \frac{A}{x}\right)$. Then $x_n = g(x_{n-1})$ defines the sequence and clearly $\sqrt{A}$ is a fixed point of g. We simply want to show that any initial, positive approximation will converge via fixed point iteration to $\sqrt{A}$. If $x_0 = \sqrt{A}$ the sequence is constant and we are done.

Suppose $x_0 \in (\sqrt{A}, \infty)$ and choose $M > \max\{x_0, A, 1\}$ so that $x_0 \in \left[\sqrt{A}, M\right]$. Note that $g'(x) = \frac{1}{2}\left(1 - \frac{A}{x^2}\right)$ is defined on $(0, \infty)$ and therefore on $(\sqrt{A}, M)$. For $\sqrt{A} < x < M$, $0 < \frac{A}{x^2} < 1$ and therefore $|g'(x)| = \frac{1}{2}\left|\left(1 - \frac{A}{x^2}\right)\right| < \frac{1}{2}(1) < \frac{1}{2}$. Also $g'(x) > 0$ on $\left[\sqrt{A}, M\right]$ so that g is increasing. Since $g(\sqrt{A}) = \sqrt{A}$ and $g(M) = \frac{1}{2}\left(M + \frac{A}{M}\right) < \frac{1}{2}(M + 1) < M$, then $g : \left[\sqrt{A}, M\right] \to \left[\sqrt{A}, M\right]$. By Theorem 2.3 (Fixed-Point Theorem), for any intial approximation $x_0 \in \left[\sqrt{A}, M\right]$, the sequence $x_n = g(x_{n-1})$ converges to the unique fixed point $\sqrt{A} \in \left[\sqrt{A}, M\right]$.

Now suppose $x_0 \in (0, \sqrt{A})$. Since $(x_0 - \sqrt{A})^2 > 0$, we do the following calculation to show that $x_1 > \sqrt{A}$ and therefore, the previous paragraph ensures convergence of $x_n = g(x_{n-1})$ to $\sqrt{A}$.

$$\begin{aligned} 0 < \frac{(x_0 - \sqrt{A})^2}{x_0} &= \frac{x_0^2 - 2x_0\sqrt{A}}{x_0} + \frac{A}{x_0} \\ &= x_0 + \frac{A}{x_0} - 2\sqrt{A} \\ &= x_1 - 2\sqrt{A} < x_1 - \sqrt{A}. \end{aligned}$$

Thus $x_1 > \sqrt{A}$.

G24) Let $g \in C^1[a, b]$ and $p \in (a, b)$ be such that $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$ then $|p_o - p| < |p_1 - p|$. This implies that a fixed point iteration following an approximation within δ of the root p will always be a worse approximation. Thus, convergence won't happen.

We know that $|g'(p)| = \lim_{p_0 \to p} \left|\frac{g(p_0) - g(p)}{p_0 - p}\right|$ and so by definition, for any $\epsilon^* > 0$ there exists a $\delta^* > 0$ such that $|p_0 - p| < \delta^*$ implies that $\left|\left|\frac{g(p_0) - g(p)}{p_0 - p}\right| - |g'(p)|\right| < \epsilon^*$.

Let $|g'(p)| = 1 + \epsilon$. From above, there is a $\delta > 0$ such that if $|p_0 - p| < \delta$ then

$$\left| \left| \frac{g(p_0) - g(p)}{p_0 - p} \right| - (1 + \epsilon) \right| < \epsilon.$$

Then

$$1 < \left| \frac{g(p_0) - g(p)}{p_0 - p} \right| < 1 + 2\epsilon$$

for all $p_0 \in [p-\delta, p+\delta]$. Mulitplying the left inequality by $|p_0 - p|$ we have that for all $p_0 \in [p-\delta, p+\delta]$, $|p_0 - p| < |g(p_0) - g(p)| = |p_1 - p|$ which establishes the claim.