

SOLUTIONS TO HOMEWORK 5

Section 7.3

6c 8c 14c (I originally assigned 10c but I really meant 14c) As usual, the solutions using Matlab appear at the end.

Section 7.4

4d) Let $A = \begin{bmatrix} .04 & .01 & -.01 \\ .2 & .5 & -.2 \\ 1 & 2 & 4 \end{bmatrix}$. Then $\|A\|_\infty = 7$. Using elementary row operations it is easy to compute $A^{-1} = \frac{1}{87} \begin{bmatrix} 2400 & -60 & 3 \\ -1000 & 170 & 6 \\ -100 & -70 & 18 \end{bmatrix}$. So we can compute $\|A^{-1}\|_\infty = \frac{2463}{87}$. Thus $K_\infty(A) = \frac{17241}{87}$.

So $\|x - \tilde{x}\|_\infty = \max\{.027586, .0151724, .065517\} = .065517$ and $K_\infty(A) \frac{\|b - A\tilde{x}\|_\infty}{\|A\|_\infty} = \frac{2463}{87} (.32) = 9.0593$.

8) In this problem $A = \begin{bmatrix} 1 & 2 \\ 1.00001 & 2 \end{bmatrix}$, $\delta A = \begin{bmatrix} 0 & 0 \\ .000001 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 3.00001 \end{bmatrix}$, and $\delta b = \begin{bmatrix} .00001 \\ .00002 \end{bmatrix}$. So the problem asks us to approximate the solution to $Ax = b$ by finding the solution to $(A + \delta A)\tilde{x} = b + \delta b$. Let \tilde{A} be the augmented matrix formed by $A + \delta A$ and $b + \delta b$. Using 7 digit rounding, Gaussian elimination results in the reduced matrix

$$\tilde{A}^{(1)} = \begin{bmatrix} 1 & 2 & 3.00001 \\ 0 & -.0000220 & -.0000120 \end{bmatrix}.$$

Backward substitution gives the approximate solution vector $\tilde{x} = (1.818182, .5909141)^t$ to the actual solution vector $x = (1, 1)^t$. We can compute $\|A\|_\infty = 3.00001$ and $\|A^{-1}\|_\infty = 200000$ so that $K_\infty(A) = 600002$. A is certainly ill conditioned.

To compute the bounds from (7.24) we also need $\|\delta A\|_\infty = .000001$ and $\|\delta b\|_\infty = .000002$. So we compute the bounds

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = \frac{.818182}{1} = .818182$$

and

$$\begin{aligned} \frac{K(A) \|A\|_\infty}{\|A\|_\infty - K(A) \|\delta A\|_\infty} \left(\frac{\|\delta b\|_\infty}{\|b\|_\infty} + \frac{\|\delta A\|_\infty}{\|A\|_\infty} \right) &= \frac{(3.00001)600002}{3.00001 - 600002(.000002)} \left(\frac{2 \times 10^{-5}}{3.00001} + \frac{10^{-6}}{3.00001} \right) \\ &= \frac{600002(.000021)}{3.00001 - 600002(.000002)} \\ &= \frac{600002(21)}{3000010 - 600002} = \frac{21}{4} = 5.25 \end{aligned}$$

with the last line obtained by multiply the numerator and denominator by 10^6 and the observation $5(600002)=3000010$. The bound is obviously a substantial overestimate to the error.

Section 7.5

14) (a) Show $\langle r^{(1)}, v^{(1)} \rangle = 0$.

$$\begin{aligned} \langle r^{(1)}, v^{(1)} \rangle &= \langle b - Ax^{(1)}, v^{(1)} \rangle \\ &= \langle b - A(x^{(0)} + t_1 v^{(1)}), v^{(1)} \rangle \\ &= \langle b - Ax^{(0)}, v^{(1)} \rangle - t_1 \langle Av^{(1)}, v^{(1)} \rangle \\ &= \langle b - Ax^{(0)}, v^{(1)} \rangle - \langle b - Ax^{(0)}, v^{(1)} \rangle = 0. \end{aligned}$$

(b) Suppose $\langle r^{(k)}, v^{(j)} \rangle = 0$ for all $1 \leq j \leq k \leq l$. Then

$$\begin{aligned} \langle r^{(l+1)}, v^{(j)} \rangle &= \langle b - Ax^{(l+1)}, v^{(j)} \rangle \\ &= \langle b - Ax^{(l)} - t_{l+1} Av^{(l+1)}, v^{(j)} \rangle \\ &= \langle b - Ax^{(l)}, v^{(j)} \rangle - t_{l+1} \langle Av^{(l+1)}, v^{(j)} \rangle \\ &= \langle r^{(l)}, v^{(j)} \rangle - t_{l+1} \langle v^{(l+1)}, Av^{(j)} \rangle. \end{aligned}$$

By the inductive hypothesis, $\langle r^{(l)}, v^{(j)} \rangle = 0$ and by A -orthogonality, $\langle v^{(l+1)}, Av^{(j)} \rangle = 0$ since $j \leq l$.

(c)

$$\begin{aligned} \langle r^{(l+1)}, v^{(l+1)} \rangle &= \langle b - Ax^{(l+1)}, v^{(l+1)} \rangle \\ &= \langle b - Ax^{(l)} - t_{l+1} Av^{(l+1)}, v^{(l+1)} \rangle \\ &= \langle b - Ax^{(l)}, v^{(l+1)} \rangle - t_{l+1} \langle Av^{(l+1)}, v^{(l+1)} \rangle. \end{aligned}$$

Since $t_{l+1} = \frac{\langle v^{(l+1)}, b - Ax^{(l)} \rangle}{\langle v^{(l+1)}, Av^{(l+1)} \rangle}$, the final term is obviously equal to 0.

By (a), (b), and (c), $\langle r^{(k)}, v^{(j)} \rangle = 0$ for all $j = 1, \dots, k$.

>> homework5_7_3

A73 =

4	1	-1	1
1	4	-1	-1
-1	-1	5	1
1	-1	1	3

b73 =

-2	-1	0	1
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x =

0	0	0	0
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6c

ans =

11 iterations

ans =

-0.7521
0.0403
-0.2803
0.6901

8c

ans =

7 iterations

ans =

-0.7532
0.0410
-0.2807
0.6916

14c

ans =

5 iterations

ans =

-0.7531
0.0412
-0.2807
0.6916

>> homework5_7_5

a =

1.0000	0.5000	0.3333
0.5000	0.3333	0.2500
0.3333	0.2500	0.2000

b =

0.8333
0.4167
0.2833

C =

1	0	0
0	1	0
0	0	1

d =

1.0000	0	0
0	1.7321	0
0	0	2.2361

Using 16 digits.

4a: Gaussian elimination

ans =

1.
-1.0000000000000003
1.0000000000000002

infinity norm of error

.2886579864025407e-14

4b: Conjugate Gradient Method

3 iterations

ans =

.9999999999993772
-1.000000000000332
.999999999997473

infinity norm of error

.6228351168147128e-12

4c: Scaled Pivoting

ans =

1.
-.999999999999951
.999999999999949

infinity norm of error

.5107025913275720e-14

4d: Preconditioned Conjugate Gradient
using d

3 iterations

ans =

1.000000000000212
-.999999999996161
1.000000000000502

infinity norm of error

.5024869409453458e-12