1. Omar Khayyam developed a technique which used the point of intersection of two conic sections in the plane to geometrically estimate solutions to cubic polynomial equations. In our class discussion, I mentioned that the following equation is actually the equation of a circle in disguise:

$$\frac{q-x}{y} = \frac{y}{x}.$$

Show that this equation is equivalent to the equation of a circle with center located at $(q/2, 0)$.

Solution:

$$\begin{aligned} \frac{q-x}{y}(xy) &= \frac{y}{x}(xy) \\ qx - x^2 &= y^2 \\ x^2 - qx + y^2 &= 0 \\ x^2 - qx + \left(\frac{q}{2}\right)^2 + y^2 &= \left(\frac{q}{2}\right)^2 \\ \left(x - \frac{q}{2}\right)^2 + y^2 &= \left(\frac{q}{2}\right)^2 \end{aligned}$$

Which is the equation of a circle of radius $q/2$, centered at $(q/2, 0)$.

2. Solve the following cubic equation in y via Tartaglia's method.

$$y^3 - 3y^2 - y + 3 = 0$$

First, *reduce* it to a cubic equation in x with no quadratic term, i.e. an equation of the form $x^3 + px + q = 0$.

(Hint 1: Set $y = x + k$, and then choose k such that the coefficient of the quadratic term is 0.)

Once you know p and q use Cardano's formula below to solve for x and finally use your values for x and k to solve for the root $y = x + k$.

(Hint 2: Cardano's formula will give you an expression involving complex numbers, but this expression simplifies to an integer.)

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \qquad \text{(Cardano's Formula)}$$

Show your work to receive full credit.

Solution:

$$\begin{aligned} (x+k)^3 - 3(x+k)^2 - (x+k) + 3 &= 0 \\ (x^3 + 3x^2k + 3xk^2 + k^3 - 3(x^2 + 2xk + k^2) - (x+k) + 3 &= 0 \\ x^3 + (3k-3)x^2 + (3k^2 - 6k - 1)x + (k^3 - 3k^2 - k + 3) &= 0 \end{aligned}$$

Let $k = 1$ then we get:

$$x^3 - 4x = 0 \quad \longrightarrow \quad p = -4,\ q = 0.$$

$$x = \sqrt[3]{-\frac{0}{2} + \sqrt{\left(\frac{0}{2}\right)^2 + \left(\frac{-4}{3}\right)^3}} + \sqrt[3]{-\frac{0}{2} - \sqrt{\left(\frac{0}{2}\right)^2 + \left(\frac{-4}{3}\right)^3}}$$

$$x = \sqrt[3]{\sqrt{(-1)^3\left(\frac{4}{3}\right)^3}} + \sqrt[3]{-\sqrt{(-1)^3\left(\frac{4}{3}\right)^3}}$$

$$x = \sqrt[3]{\sqrt{(-1)^3\left(\frac{4}{3}\right)^3}} - \sqrt[3]{\sqrt{(-1)^3\left(\frac{4}{3}\right)^3}}$$

$$x = 0.$$

Thus $y = x + k = 0 + 1 = 1$.