Name: ______________________

1. Consider the forced, linear system,

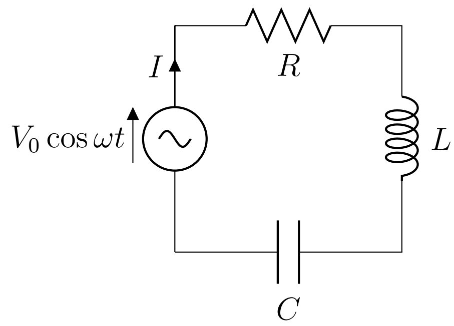
$$y^{(4)} + 2y^{(3)} + y'' = 6x. \tag{1}$$

   (a) Find the homogeneous solution y_h.

   (b) Find a particular solution y_p.
   Hint: Your trial particular solution should be $y_p = Ax^2 + Bx^3$.

   (c) Find the general solution $y(x) = y_h + y_p$.

2. Consider the following series RLC circuit:



Using Kirchoff's law for closed loops yields the following differential equation which governs the charge on the capacitor, $q(t)$.

$$L\, q'' + R\, q' + \frac{1}{C}\, q = V_0 \cos \omega t \tag{2}$$

Assume initial conditions: $q(0) = 0,\ q'(0) = 0$, and the following values for L, R and C:

$$\begin{aligned} L &= 1 \text{ H (Henry)}, \\ R &= 0\ \Omega,\ \text{(Ohm)}, \\ C &= \frac{25}{9} \text{ F (Farad)}, \\ V_0 &= 3 \text{ V (Volt)}. \end{aligned}$$

(a) What is the natural angular frequency, ω_0 of this system?
Hint: Recall ω_0 is the angular frequency of the solution to the unforced (associated homogeneous) equation:

$$L\, q'' + R\, q' + \frac{1}{C}\, q = 0.$$

(b) Assume $\omega \neq \omega_0$. Use the *method of undetermined coefficients* to solve for a particular solution, q_p of equation (2). Finally, use the general solution $q(t) = q_h + q_p$ to solve the IVP.

(c) Write down the specific case of the solution $q(t)$ (found in part b) for $\omega = 0.5$. Compute the period ($T = \frac{2\pi}{\omega}$) of this solution, which is a superposition of two cosine functions. Use MATLAB or Maple to graph one period of the solution. What phenomenon is exhibited by this solution?

(d) Now let $\omega = \omega_0$. Use the *method of undetermined coefficients* to solve for a new particular solution q_p. Then use $q(t) = q_h + q_p$ to solve the initial value problem. Graph this solution over the interval $0 \leq t \leq 60$ seconds. What phenomenon does this solution exhibit?

3. Consider the same RLC circuit as in problem 2. For this problem, take $L = 1$ H, $R = 1.2\ \Omega$, $C^{-1} = 0.36\ \mathrm{V} \cdot \mathrm{C}^{-1}$, $V_0 = 3$ V, and $\omega = 1$. This gives us the differential equation:

$$q'' + 1.2q' + 0.36q = 3\cos t. \tag{3}$$

   (a) Use the *method of undetermined coefficients* to find a particular solution $q_p(t)$ to this differential equation.

   (b) Use the particular solution q_p found in part (a) and the solution q_h to the corresponding homogeneous equation to write down the general solution to this differential equation. Identify the "steady periodic" and "transient" parts of this general solution.

   (c) Given the initial conditions $q(0) = 0$, $q'(0) = 0$, solve the resulting IVP for $q(t)$ satisfying the differential equation at the beginning of this problem (you may use Maple).

   (d) Graph, on a single plot, the solution to the IVP in part (c) as well as the steady periodic solution identified in part (b). Choose a time interval so that you can clearly see the convergence of the IVP solution to the steady periodic solution.

4. The *Laplace transform* is an operator $\mathscr{L}$, which takes as input a function of time, $f(t)$ and outputs a function of frequency, $F(s)$ according to the rule:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ \mathrm{d}t \tag{4}$$

Transform methods are very useful for solving differential equations, but a physical relationship between $F(s)$ and $f(t)$ is not immediately clear. The goal of this problem is to give intuition for the variable s.

(a) Consider the following functions:

$$f_1(t) = \begin{cases} -1 & 0 \le t \le 1 \\ +1 & 1 < t \end{cases}$$
$$f_2(t) = \cos 2t$$
$$f_3(t) = \sin 3t$$

Find the Laplace transforms $F_1(s), F_2(s), F_3(s)$ of these functions. It should be straightforward to compute $F_1(s)$ via hand by breaking up the integral into two integrals, one with bounds 0 and 1, and the other with bounds 1 and ∞. You can just look up $F_2(s)$ and $F_3(s)$ using the table in your textbook.

(b) Rewrite each of the Laplace transforms found in part (a), by replacing the variable s with $i\omega$. Here i is the imaginary number $i^2 = -1$, and ω is a real number which represents an angular frequency.

(c) If $z = x + iy$ is a complex number, its *complex conjugate* is $z^* = x - iy$. The non–negative real number $|z| = \sqrt{zz^*}$ is called the *magnitude* of z. Find the magnitude of $F_1(i\omega), F_2(i\omega)$, and $F_3(i\omega)$. *Hint:* For this problem, the complex conjugate of $F(i\omega)$ is $F(-i\omega)$, and $e^{\pm ix} = \cos x \pm i \sin x$.

(d) Plot the magnitudes $|F_1(i\omega)|, |F_2(i\omega)|$, and $|F_3(i\omega)|$. Can you relate these plots to the frequency of $f_2(t)$ and $f_3(t)$? Based on that relationship, what can you say about the frequencies of $f_1(t)$?