

Name: _____

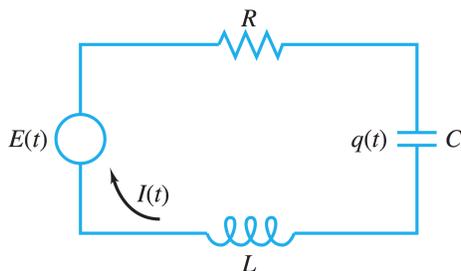
1. This problem deals with a highly simplified model of a car of weight 3200 lb (mass $m = 100$ slugs in fps units). Assume that the suspension system acts like a single spring and its shock absorbers like a single dashpot, so that its vertical vibrations satisfy

$$mx'' + cx' + kx = 0, \quad (1)$$

with appropriate values of the coefficients.

- (a) Find the stiffness coefficient k of the spring if the car undergoes free vibrations at 80 cycles per minute (cycles/min) when its shock absorbers are disconnected, that is when $c = 0$.
- (b) With the shock absorbers connected the car is set into vibration by driving it over a bump, and the resulting damped vibrations have a frequency of 78 cycles/min. After how long will the time-varying amplitude be 1% of its initial value?

Series RLC Circuits



A series RLC electrical circuit is exactly analogous to a mechanical mass–spring–dashpot system. To generate the equation that governs a mechanical system we relied upon Newton’s second law, which in its most general form could be stated as momentum cannot be created or destroyed, i.e. momentum is conserved.

To generate the equation that governs the flow of charge in the above electrical circuit we use Kirchoff’s voltage law: the voltage changes around any closed loop sum to zero. This is another conservation law. The voltage changes due to each circuit element are:

- $V_C = -\frac{1}{C}q$,
- $V_R = -RI = -Rq'$,
- $V_L = -LI' = -Lq''$.
- $V_E = E(t)$.

Thus we get $V_E + V_R + V_C + V_L = 0$, which yields the initial value problem:

$$LI' + RI + \frac{1}{C}q = E(t) \quad q(0) = q_0 \quad I(0) = I_0. \quad (2)$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, $q(t)$ is the charge in coulombs on the capacitor at time t , and $I = q'$ is the current in amperes.

Mechanical System	Electrical RLC Series Circuit		
$mx'' + cx' + kx = F(t)$		$Lq'' + Rq' + \frac{1}{C}q = E(t)$	
Displacement	x	Charge	q
Velocity	x'	Current	$q' = I$
Acceleration	x''	Time rate of change of current	$q'' = I'$
Mass	m	Inductance	L
Damping constant	c	Resistance	R
Spring constant	k	Conductance = (Capacitance) ⁻¹	$1/C$
External force	$F(t)$	Voltage source (battery)	$E(t)$

2. Find the current at time t if the charge on the capacitor is initially zero, $L = 10$ H, $R = 20\Omega$, $C = (6260)^{-1}$ F and $E(t) = 100$ V.

Hint: Differentiate both sides of the equation to obtain a homogeneous linear second order equation for $I(t)$. Then use equation (2) to determine dI/dt at $t = 0$.

Energy in a Mass–Spring–Damper System

3. Let $x(t)$ be the position of a mass m attached to a spring with Hooke's constant k and damping piston with constant c , yielding the differential equation

$$mx'' + cx' + kx = f(t),$$

where $f(t)$ is an external forcing on the mass. We wish to account for the total energy of the mass-spring configuration, neglecting the heat energy loss due to damping. We define the total energy $E(t)$ to be the sum of kinetic and potential energy. Potential energy $PE(t)$ is stored by the compressed or stretched spring, and is the work done to stretch/compress it as the mass moves from equilibrium $x = 0$, to position x :

$$PE(t) = \int_0^x kudu = \frac{1}{2}kx^2.$$

As usual, the kinetic energy of the mass is

$$KE(t) = \frac{1}{2}m(x')^2.$$

The sum is the total energy

$$E(t) = PE(t) + KE(t) = \frac{1}{2}(kx^2 + m(x')^2).$$

- Take the derivative of $E(t)$ with respect to time. Use the chain rule on the right side of the equation. Then simplify your result, so that you get a formula for $E'(t)$ that only depends on the forcing $f(t)$, the velocity $x'(t)$, and the damping coefficient c .
- What conditions on m, c, k guarantee that the energy in the system is constant (i.e., $dE/dt = 0$), assuming that there is no external forcing, $f = 0$?
- Set $f(t) = 0$, $m = 1$, $c = 2$, and $k = 2$. Set initial conditions to be $x(0) = 0$ and $x'(0) = 1$. How long will it take for the system to lose 80% of its initial energy? To solve, find the solution $x(t)$ to the DE and plug into the energy function E definition to find an explicit formula. The double angle identity might be useful: $\sin(2u) = 2\sin(u)\cos(u)$. Plot the energy curve $E(t)$ and describe its behavior. Does the energy decrease monotonically, or is there an oscillation? Explain in words the mechanism that drives the energy picture that you observe.

4. Let y_p be a solution to the nonhomogeneous, linear equation

$$y'' + a(x)y' + b(x)y = f(x), \quad (3)$$

and let y_1 and y_2 be solutions to the associated homogeneous equation

$$y'' + a(x)y' + b(x)y = 0. \quad (4)$$

Show that $y = c_1y_1 + c_2y_2 + y_p$, where c_1 and c_2 are scalars, is a solution to equation (3).