Name: ______________________________

1. Suppose that A is an $n \times n$ matrix and that λ is a constant real number. Define W to be the set of all vectors $\vec{x}$ that satisfy the equation

$$A\vec{x} = \lambda\vec{x}.$$

   (a) Show that the set W is a subspace of $\mathbb{R}^n$. (Hint: §4.2 Theorem 1.)

   (b) Let $\lambda = 1$ and let A be given by

$$A = \begin{bmatrix} -4 & 1 & 1 & 1 \\ -16 & 3 & 4 & 4 \\ -7 & 2 & 2 & 1 \\ -11 & 1 & 3 & 4 \end{bmatrix}$$

   Find a basis for the subspace W, the set of solutions to $A\vec{x} = \lambda\vec{x}$.

   Hint 1: Theorem 2 of §4.2 says that the set of solutions to systems of the form $B\vec{x} = \vec{0}$ form subspaces. Use the rules of matrix algebra to transform $A\vec{x} = \vec{x}$ into the form $B\vec{x} = \vec{0}$, then solve for the solution space (via the algorithm in §4.4 p. 258) and find a basis for the solution space of the transformed equation $B\vec{x} = \vec{0}$.

   Hint 2: The fact that $I\vec{x} = \vec{x}$, where I is the 4×4 identity matrix, is useful.

   Hint 3: Use either Maple or MATLAB to compute RREF of matrix B.

   - Maple: `LinearAlgebra[ReducedRowEchelonForm](B)`
   - MATLAB: `rref(B)`

2. In §4.7 we learned that the set of polynomials of degree n form a subspace of the vector space of functions of a real variable, $\mathcal{F}$. Let P be given by:

$$P = \{1, x, 3x^2 - 1, 5x^3 - 3x\}$$

   (a) Determine if the set P is linearly independent or linearly dependent. (Hint: The Wronskian is a useful tool.)

   (b) Find a linear combination of elements from P that generates the polynomial $y(x) = 1 + x + x^2 + x^3$.

   (c) Does P span the vector space V of all polynomials of degree three or less? Justify your answer.

3. We define the linear DE for $x > 0$:

$$x^2y'' + axy' + by = 0.$$

Use the substitution $u = \ln(x)$ to transform the equation to the constant-coefficient equation of the form

$$\frac{d^2y}{du^2} + a\frac{dy}{du} + by = 0.$$

(a) Find a, b, c. Hint: set $x = e^u$ and take derivatives of y with respect to u use the chain rule multiple times to obtain y'' and y' in terms of the new variable u.

(b) Solve the constant-coefficient equation in u to find two linearly independent solutions. Hint: You should have 2 solutions.

(c) Use part (b) to find solutions of the DE when $a = 1$ and $b = -\frac{1}{2}$. Find the solution when $y(1) = 0$ and $y'(1) = 1$.

4. Consider the differential equation: $mx'' + kx = 0$.
   (a) Verify that $\cos(\omega t)$ and $\sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$, are linearly independent solutions to the DE.
   (b) Solve the DE with initial conditions: $y(0) = 1$ and $y'(0) = 0$.
   (c) Show that $\sin(\omega t)$ and $\sin(\omega t + \pi)$ linearly independent solutions of the DE.
   (d) Use $\sin(\omega t)$ and $\sin(\omega t + \pi)$ to solve the IVP from part b.
   (e) Double check that your solution from parts b and d are equivalent using the angle-sum identity.