

Name: \_\_\_\_\_

1. The following application of Linear Algebra to curve fitting is explained in greater detail in section §3.7 of your text (page 218).
  - (a) Determine the equation of a line passing through the points  $(-1,-2)$  and  $(3,2)$  in the following way. Assume the line has equational form

$$y = c_0 + c_1x, \tag{1}$$

and put both data points into equation (1) to get two equations in the two unknowns  $c_0$  and  $c_1$ . Write this system of equations as a matrix equation of the form:  $A\vec{x} = \vec{b}$ . Now use any linear algebra technique you like (row reduction on the augmented matrix, Cramer's rule, or compute  $A^{-1}\vec{b}$ ) to solve for the unknowns  $\vec{x} = [c_0 \ c_1]^T$ .

- (b) Now use the same technique to find the equation of the parabola which passes through the points:  $(3,1)$ ,  $(2,3)$ ,  $(0,-5)$ , but this time you will need to assume that the equation has the form:

$$y = c_0 + c_1x + c_2x^2, \quad (2)$$

and you will generate three equations in the three unknowns:  $c_0, c_1, c_2$ . Write this system of equations as a matrix equation of the form  $A\vec{x} = \vec{b}$  and solve for  $\vec{x} = [c_0 \ c_1 \ c_2]^T$  using the technique of your choice.

2. Find all  $\alpha$  such that  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  are linearly dependent. (See §4.1 Theorem 4, page 233.)

$$\vec{a}_1 = \begin{bmatrix} \alpha \\ 2 \\ -2 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ \alpha \\ 2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -2 \\ 2 \\ \alpha \end{bmatrix}.$$

3. This question demonstrates a geometric interpretation of the determinant.

- (a) Suppose we have a matrix  $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$  and a unit square determined by the points:  $(0,0)$ ,  $(0,1)$ ,  $(1,1)$ ,  $(1,0)$ . What will the image of the square look like under transformation by the matrix  $A$ ? i.e.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where  $(x_1, x_2)$  is a point on the original square and  $(y_1, y_2)$  is its image after transformation by  $A$ . Draw the image of the square after this transformation.

- (b) Find the area of the parallelogram in (a). (Hint: think of the parallelogram as sitting inside a larger rectangle  $0 \leq y_1 \leq 4, 0 \leq y_2 \leq 4$ ). Compare this area to the determinant of  $A$ .

- (c) Show, for the general  $\vec{u}, \vec{v}$  in the first quadrant, with  $\vec{v}$  counterclockwise from  $\vec{u}$ , the area of the parallelogram having the vectors  $\vec{u}, \vec{v}$  as adjacent sides (as in the special case above) always equals  $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$ .
- (d) What happens if you compute  $\begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix}$  instead?

4. Vector spaces are ubiquitous in math, science and engineering. This problem gives an example of an unusual vector space. Let  $V$  be the set of all real numbers, so our vectors are just the familiar real numbers. However, vector addition and scalar multiplication are defined differently from normal addition and multiplication. To distinguish between the usual real number addition and the new vector addition we will use the symbol  $\oplus$  to represent vector addition. The definition of vector addition is:

$$x \oplus y = x + y + 7.$$

Thus to add two numbers as vectors, you add them in the normal way but then also add 7. So for example,  $3 \oplus 5 = 3 + 5 + 7 = 15$ .

To distinguish scalar multiplication we will use the symbol  $\circ$ . It is defined this way:

$$r \circ x = rx + 7(r - 1).$$

That is, if  $r$  is a scalar, and if  $x$  is thought of as a vector in our system, then the scalar multiplication combines them to form  $rx$  multiplied in the usual way, plus  $7(r - 1)$  also computed in the usual way. For example, if  $r = 3$  and  $x = 5$ , then

$$r \circ x = 3 \circ 5 = 15 + 7(3 - 1) = 29.$$

- (a) Show that vector addition is commutative. That is show:

$$x \oplus y = y \oplus x.$$

- (b) Show that vector addition is associative. That is, show:

$$(x \oplus y) \oplus z = x \oplus (y \oplus z).$$

- (c) What real number plays the role of the zero vector (additive identity) in this vector space? Hint: it is *not* 0.

- (d) What is the additive inverse of the vector 8?
- (e) Notice that 1 is still the multiplicative identity because  $1 \circ x = 1x + 7(1 - 1) = x$ . How does one undo scaling by 3? Hint: make the obvious guess and check to see if it is correct. That is, if your guess is  $r$ , then check whether  $r \circ (3 \circ x) = x$ .
- (f) Use your answers from parts (d) and (e) to solve for  $x$  in:  $8 \oplus (3 \circ x) = 28$ . You could solve this by just guessing values for  $x$ , but that will not suffice. You must show how to solve the equation algebraically by using the concepts of additive inverse, additive identity, and associativity.